THE
ABEL
PRIZE
2019

The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2019 to

Karen Keskulla Uhlenbeck

University of Texas at Austin

“for her pioneering achievements in geometric partial differential equations, gauge theory and integrable systems, and for the fundamental impact of her work on analysis, geometry and mathematical physics.”

Karen Keskulla Uhlenbeck is a founder of modern Geometric Analysis. Her perspective has permeated the field and led to some of the most dramatic advances in mathematics in the last 40 years.

Geometric analysis is a field of mathematics where techniques of analysis and differential equations are weaved with the study of geometrical and topological problems. Specifically, one studies objects such as curves, surfaces, connections and fields which are critical points of functionals representing geometric quantities such as energy and volume. For example, minimal surfaces are critical points of the area and harmonic maps are critical points of the Dirichlet energy. Uhlenbeck’s major contributions include foundational results on minimal surfaces and harmonic maps, Yang-Mills theory, and integrable systems.

Minimal surfaces and bubbling analysis

An important tool in global analysis, preceding the work of Uhlenbeck, is the Palais— Smale compactness condition. This condition, inspired by earlier work of Morse, guarantees existence of minimisers of geometric functionals and is successful in the case of 1-dimensional domains, such as closed geodesics.

Uhlenbeck realised that the condition of Palais—Smale fails in the case of surfaces due to topological reasons. The papers of Uhlenbeck, co-authored with Sacks, on the energy functional for maps of surfaces into a Riemannian manifold, have been extremely influential and describe in detail what happens when the Palais-Smale condition is violated. A minimising sequence of mappings converges outside a finite set of singular points and by using rescaling arguments, they describe the behaviour near the singularities as *bubbles* or *instantons*, which are the standard solutions of the minimising map from the 2-sphere to the target manifold.

In higher dimensions, Uhlenbeck in collaboration with Schoen wrote two foundational papers on minimising harmonic maps. They gave a profound understanding of singularities of solutions of non-linear elliptic partial differential equations. The singular set, which in the case of surfaces consists only of isolated points, is in higher dimensions replaced by a set of codimension 3.

The methods used in these revolutionary papers are now in the standard toolbox of every geometer and analyst. They have been applied with great success in many other partial differential equations and geometric contexts. In particular, the bubbling phenomenon appears in many works in partial differential equations, in the study of the Yamabe problem, in Gromov's work on pseudoholomorphic curves, and also in physical applications of instantons, especially in string theory.

Gauge theory and Yang-Mills equations

After hearing a talk by Atiyah in Chicago, Uhlenbeck became interested in gauge theory. She pioneered the study of Yang-Mills equations from a rigorous analytical point of view. Her work formed a base of all subsequent research in the area of gauge theory.

Gauge theory involves an auxiliary vector bundle over a Riemannian manifold. The basic objects of study are connections on this vector bundle. After a choice of a trivialisation (gauge), a connection can be described by a matrix valued 1-form. Yang-Mills connections are critical points of gauge-invariant functionals. Uhlenbeck addressed and solved the fundamental question of expressing Yang-Mills equations as an elliptic system, using the so-called Coulomb gauge. This was the starting point for both Uhlenbeck's celebrated compactness theorem for connections with curvature bounded in L^p, and for her later results on removable singularities for Yang-Mills equations defined on punctured 4-dimensional balls. The removable singularity theory for Yang-Mills equations in higher dimensions was carried out much later by Gang Tian and Terence Tao. Uhlenbeck's compactness theorem was crucial in Non-Abelian Hodge Theory and, in particular, in the proof of the properness of Hitchin's map and Corlette's important result on the existence of equivariant harmonic mappings.

Another major result of Uhlenbeck is her joint work with Yau on the existence of Hermitian Yang-Mills connections on stable holomorphic vector bundles over complex n-manifolds, generalising an earlier result of Donaldson on complex surfaces. This result of Donaldson-Uhlenbeck-Yau links developments in differential geometry and algebraic geometry, and is a foundational result for applications of heterotic strings to particle physics.

Uhlenbeck's ideas laid the analytic foundations for the application of gauge theory to geometry and topology, to the important work of Taubes on the gluing of self-dual 4manifolds, to the ground-breaking work of Donaldson on gauge theory and 4dimensional topology, and many other works in this area. The book written by Uhlenbeck and Dan Freed on "Instantons and 4-Manifold Topology" instructed and inspired a generation of differential geometers. She continued to work in this area, and in particular had an important result with Lesley Sibner and Robert Sibner on non self-dual solutions to the Yang-Mills equations.

Integrable systems and harmonic mappings

The study of integrable systems has its roots in 19th century classical mechanics. Using the language of gauge theory, Uhlenbeck and Hitchin realised that harmonic mappings from surfaces to homogeneous spaces come in 1-dimensional parametrised families. Based on this observation, Uhlenbeck described algebraically harmonic mappings from spheres into Grasmannians relating them to an infinite dimensional integrable system and Virasoro actions. This seminal work led to a series of further foundational papers by Uhlenbeck and Chuu-Lian Terng on the subject and the creation of an active and fruitful school.

The impact of Uhlenbeck's pivotal work goes beyond geometric analysis. A highly influential early article was devoted to the study of regularity theory of a system of non-linear elliptic equations, relevant to the study of the critical map of higher order energy functionals between Riemannian manifolds. This work extends previous results by Nash, De Giorgi and Moser on regularity of solutions of single non-linear equations to solutions of systems.

Karen Uhlenbeck's pioneering results have had fundamental impact on contemporary analysis, geometry and mathematical physics, and her ideas and leadership have transformed the mathematical landscape as a whole.