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Chess, chaos and a 'know-it-all' demon

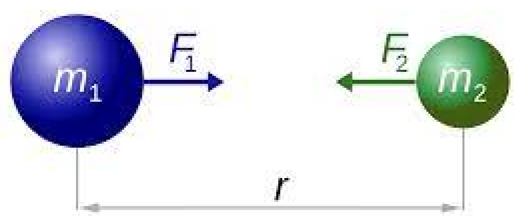
How random is a coin? I remember being upset to learn, during GCSEs, that true *randomness* arises from our inability to predict something - that fluctuations in the Gieger Muller counter, measuring radiative decay were truly random (and spontaneous), but that my trusty coin flip was demoted to 'pseudo-random'.

This gray area, things that seem random but aren't, is all to do with chaos. To be very clear, chaos is not randomness, yet there arise issues in treating chaotic systems as deterministic in any practical sense; they exist in a liminal middle ground. To an omniscient god, a coin toss is entirely deterministic. Mathematician Peirre-Simon Laplace talked about such a being - Laplace's demon, who could accurately predict the outcome a coin toss would yield, familiar with every mechanism within the interaction between my hand and the coin. Comparatively, I rely on heuristics - a 'sense' of what the mechanisms taking place.This coin flip is analogous to chaos, an interaction that is incredibly sensitive to initial conditions, and thus seemingly random given imprecision within measurements.

Chaos is difficult to define - described by its mathematical appearance, rather than an inherent quality. Notice the difference in defining a quadratic function, whose definition is generally based on the presence of a second degree polynomial. Instead, if I could say 'a function which yields outputs that increase with increasing rate - generating a parabolic graph' -I'd say the clear difference is the second definition doesn't prescribe any sufficient conditions, such that we are able to construct our own quadratic functions - rather it tells us how we can spot a quadratic function based on sets of outputs, or shapes of plotted graphs - functioning as a *descriptive* definition rather than *prescriptive*. This descriptive definition of chaos makes it incredibly difficult to understand its *ingredients*.

Arguably, chaos has been best described by Lorenz, who summarised it as 'When the present determines the future, but the approximate present does not approximately determine the future.' It's a feature we constantly see - in nature (as with the weather) and artificial constructs (such as the stock market).

An early example of chaos was observed by Henri Poincare; the n-body problem: a development on the two-body problem - where physicists concerned themselves with the motion of two gravitating masses. This two-body problem can be solved by treating it as individual one-body problems. We can use Newton's law of universal gravitation and Newton's second law, to find the force exerted on the two masses and therefore the acceleration. Through calculus, we are able to leverage this information on acceleration to clarify the relationship between velocity and time - enabling us to predict future velocities with accuracy and ease. Similarly, velocity informs us of position regarding time.



$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

$$\mathbf{F}_{12}(\mathbf{x}_1, \mathbf{x}_2) = m_1 \ddot{\mathbf{x}}_1 \qquad \text{(Equation 1)}$$

$$\mathbf{F}_{21}(\mathbf{x}_1, \mathbf{x}_2) = m_2 \ddot{\mathbf{x}}_2 \qquad \text{(Equation 2)}$$

With that being said, the two body problem is solved through many clever deductions that drastically reduce the amount of unknowns. For example we can consider one center of mass - a *Barycenter* which doesn't accelerate. This allows us to consider the relative positions about the centre of mass with ease. Visually, there's a neat symmetry within the two-body problem, since Newton's third law states the forces on both masses are equal and their relative distance is constant. As we move towards any n-body problem, we run into indeterminate systems, where we have too many unknowns, rendering the system unsolvable, as Poincare proved. Furthermore, the behavior of these systems is chaotic. Intuitively, this sudden leap in the sensitivity of initial conditions makes sense since there isn't much symmetry. Slight differences in positions or velocities of the masses may lead to a slightly stronger attractive force, which leads to the distance of separation reducing more than it otherwise would and the attractive force getting larger, which leads to the distance of separation being reduced more - a chain reaction which amplifies this slight distance to a large increase in acceleration. This increase may lead to the mass being flung towards the third mass, having such high kinetic energy that it shoots past into a far-off space - as often does happen with minor adjustments to simulations. I encourage you to tinker, altering the positions and velocities within this brilliantly simple simulation. You'll notice that, as a default, the simulation demonstrates what is known as a 'periodic solution' of the three body problem; these solutions produce easily predictable periodic patterns - yet they are very much 'special cases'. This behavior isn't simply unique to the three body problem, in investigating chaos in many different fields, there exist *fixed points* and *limit cycles*, values which do lead to patterns; in the case of a fixed point it could be considered a pattern of period 1. Lagrange in particular, found a symmetrically satisfying family of periodic solutions where the three masses form an equilateral triangle at each instant.

To understand some of the features of chaos, simplifying the problem helps. The first thing we should ask is how sensitivity arises? What seemingly magical interaction blows up tiny uncertainties in certain functions but not others? Often it is to do with the one-of-a-kind nature of the exponential function. The true magic of exponentials is well illustrated in the story of the invention of chess: an Indian king was absolutely mesmerised by the game of chess and wanted to reward its inventor, who, when asked what he would like, said that he would like one grain of wheat to be placed on the first square of the chess board, two grains of wheat to be placed on the second, four on the third, and so on - (doubling the number of grains on each subsequent square); once finished, he would like to receive that much wheat. The king laughed, baffled by such a meagre request - at least a meagre request in *his eyes*. In actuality, when the king's advisors calculated this sum, they realised that this was no small request. The total number of grains, is greater than the total number of wheat grains produced in the past two millennia.

$$\sum_{i=0}^{63} 2^i.$$

Exponential functions' growth are ones that are deceptively slow for small initial values, but overtake linear functions quickly; at some point, regardless of the parameters, exponential functions will overtake any function expressed as $y = x^n$, n being any power. Errors within the function grow at exorbitant rates. Let's assume that the inventor asked for 11/10ths of a grain from the start, instead of 1, but was rounded by the lazy advisors. Whilst the sum total is:

$$\sum_{i=0}^{63} 1 * 2^i$$

It really should be:

$$\sum_{i=0}^{63} 1.1 * 2^{i}$$

This is a tiny change; our inventor wouldn't notice the missing tenth of a grain of wheat. However, when checking the fourth tile, he'll realise he's missing 1.6 grains. This is a 16x increase in the difference between what the inventor asked for, and the value defined by a tiny human error. On the final tile, the difference between the inventor's expectation and the actual number is about 200x the amount of wheat produced globally: $4.61 * 10^{-17}$ grains - that's one ticked off inventor!

This ridiculous example is entirely relevant towards our conversations about chaos. If we were making predictions about, say, the transmission of certain diseases amongst a population; rounding errors (much as the advisors did in treating 1.1 as 1) could yield massively different consequences, purely in terms of the rate of growth.

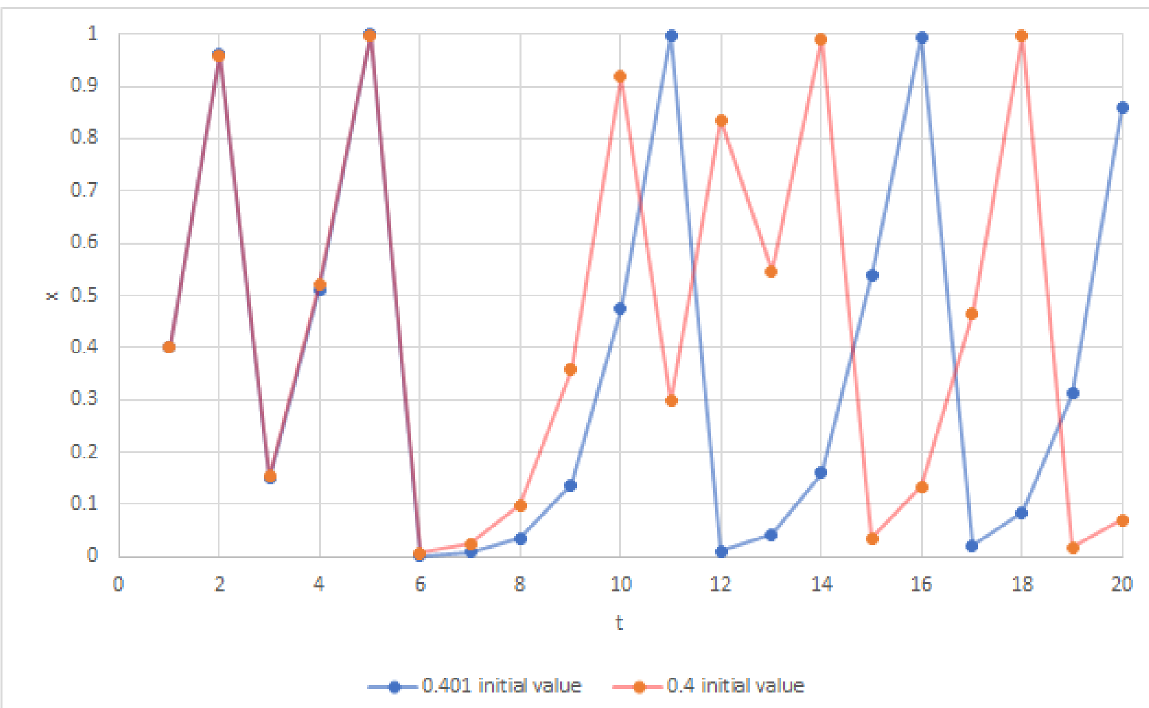
Consider this, if we are to think about the progress along the X axis as time, regardless of how small the difference between the two initial states are - the two lines will diverge such that their separation tends towards infinity. Essentially, even a minutely imprecise measurement of any initial state, will yield results that are completely different to the actual values. It's as if every measurement has an *expiry date*,when the data becomes useless for forecasting. More formally, we can measure the amount of time it takes for the distance between the two nearby trajectories to increase by a factor of *e* - the Lapunov time.
However, when we talk about chaos, we don't usually refer to exponentials, since we have a basic intuition that they shoot up towards infinity at a rapid rate. For us, the 'pseudo-randomness' of chaos comes from an inability to tell any sort of direction to the function - one that winds around, back and forth. There is an element of containment within chaos - in jumping up and down we see boundaries - physical limitations or mathematical constraints. It should be clear in the next example, that chaos must have an additional property of recurrence in order to be contained. Recurrence need not be immediate nor does the sequence need to reach any arbitrary recurrence point we choose, but as long as it is able to come closer to that point than any previous value at some point in the future, the sequence can be defined as recurrent.

If we were to make a simple model for chaos, we need a function which has both recurrence and sensitivity. We can have a term-to-term rule like such:

$$U_{n+1} = 4 * U_n * (1 - U_n)$$

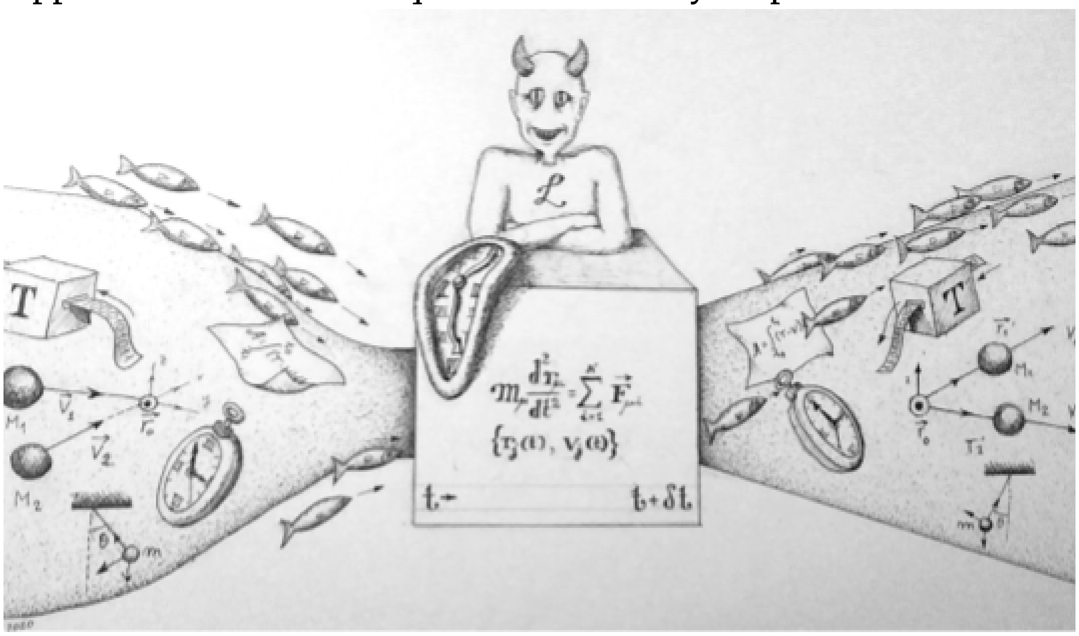
If we are to ignore the (1- U_n), this is simply a term-to-term rule for an exponential sequence, which is exceptionally sensitive. We have also introduced a component (1-U_n). With U being within 0 and 1, this component will reduce the total product, however, crucially, it will do so more for values of U_n closest to 1 and less for values closest to 0. Therefore, for $1-U_n > 0.25$, or $U_n < 0.75$ the function will grow; if $U_n > 0.75$ the function will fall. This growth and fall relationship creates recurrence, making this function, known as the logistic map function, one of the simplest examples of chaos. There is far more to the logistic map function - I haven't talked about the key role the parameter plays.

We can see the chaotic effects of this function by looking at graphs for the first twenty terms of a sequence defined by the logistic map function starting on 0.401 and 0.4:



For values of t (the order of terms within the sequence), up until t = 8, the sequences have similar shapes. However, from there, notice the unique shapes we see within the sequences - the 'M' shape formed within t = 12 to t = 15 -not found within the sequence which has an input value only a thousandth greater, emphasising the incredible sensitivity of this function.
Patterns like such are often seen in nature; where both negative and positive

feedback loops lead to booms in populations followed by steep declines in the population size. Looking at the two sequences, you can recognise that, without doing the calculation, it would be purely speculative to guess the fifth term down at any point in the sequence would be. Even with the information about ‘the 0.4 line’ and ‘the 0.401 line’; predicting the shape of the 0.402 line for twenty terms without doing the calculations would be impossible - as Lorenz aptly described, slight approximations in initial values don’t correspond to slight approximations in subsequent values in any shape or form.



Laplace's demon

These problems shape the world around us, limiting our ability to forecast and causing a great amount of suffering from it. Whether it is the volatility of the stock market, leaving many in ruins, or a falsely sunny weather forecast that leaves me drenched - so many systems that our lives are dictated by are constantly jumping up and down in chaos. Chaos limits our eyesight in science, severely undermining the role of simulation in long term predictions. It seems to be an unshakeable curse; there will always be another decimal place that our instruments are not precise enough to cover, like walking on a treadmill - each step we take brings us nowhere closer to the exact value - if one even exists. Yet I would argue it is a blessing too - I imagine Laplace's demon, if they were to exist, would suffer greatly in being able to determine the fate of the universe until the end of time. In understanding every law and knowing every unknown, can we even distinguish between the future and past?

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