The Euler-Lagrange Equation

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1 Introduction

The Euler-Lagrange equation is a powerful equation capable of solving a wide variety of optimisation problems that have applications in mathematics, physics and engineering. Problems such as finding the optimal path between 2 points or finding the optimal shape of a projectile or even developing new physical theories are formulated and heavily rely upon the Euler-Lagrange equation.

We are all familiar with the standard way of minimising a function: differentiate, set to 0 and solve whilst using the second derivative to verify you are dealing with a minimum. These optimization problems, however, deal with objects known as functionals. Functionals are roughly equivalent to “functions of functions”. They map the set of functions to a set of scalar values. Loosely speaking, an ordinary function can be thought of as a list of infinitely many values (each representing the function evaluated at different points) and thus functionals can be treated as functions of infinitely many independent variables.

Here, our standard ways of finding extrema are incapable of dealing with so much information. This is where the Euler-Lagrange equation comes into play, as this equation is well-equipped to deal with these optimisation problems.

Finding the shortest path between 2 points is an example of an optimisation problem that can be solved using the Euler-Lagrange equation. The functional here is distance: it maps a path to its length. To find the distance along a general path, you have to apply pythagoras’ theorem infinitely many times on infinitesimal segments that resemble straight lines enough that very accurate approximations can be made.
The length of the curve, $S$, can then be estimated by summing the lengths of these small line segments.

$$
S = \sum_{i=0}^{n} \sqrt{dx_i^2 + dy_i^2}
= \sum_{i=0}^{n} \sqrt{1 + \left(\frac{dy_i}{dx_i}\right)^2}dx_i
$$

In the limit of $n$ to infinity this sum becomes an integral and it is the integrand that we call the lagrangian. The integral is from $a$ to $b$ which represent the start and end points of the curve. The lagrangian is denoted by $L$.

$$
S = \int_{a}^{b} \sqrt{1 + (y')^2}dx
$$

In general, the lagrangian is a function that depends on the local nature of a specific point on the function (the value of the function and its derivative at that point) that gives you a measure of how “bad” that portion of the function is. This is why the lagrangian must be integrated over the entire range to compute the total value of the functional.

2 Euler-Lagrange equation

Now that we have an intuitive understanding of functionals and lagrangians we can finally have a look at the Euler-Lagrange equation. Let $S$ be a functional such that:

$$
S = \int_{a}^{b} L(x, y(x), y' )dx
$$

Any stationary function of $S$ (a function that minimises/maximises $S$) obeys the following equation:

$$
\frac{\partial L}{\partial y} = \frac{d}{dx} \frac{\partial L}{\partial y'}
$$
If $L$ has no explicit dependence on $x$ then the equation can be simplified to:

$$\Rightarrow L - y' \frac{\partial L}{\partial y'} = K$$

where $K$ is an arbitrary constant.

A proof of these equations is provided in the appendix.

3 Proof that the shortest path between 2 points is a straight line

Now we will use the Euler-Lagrange equation to find the shortest path between two points. We have already shown that the relevant lagrangian for this problem is:

$$L = \sqrt{1 + (y')^2}$$

Since there is no explicit dependence on $x$ we can use the simplified version of the Euler-Lagrange equation.

$$\sqrt{1 + (y')^2} - y' \frac{\partial}{\partial y'} \sqrt{1 + (y')^2} = K$$

$$\Rightarrow \sqrt{1 + (y')^2} - \frac{(y')^2}{\sqrt{1 + (y')^2}} = K$$

Multiplying through by $\sqrt{1 + (y')^2}$ gives:

$$1 = C \sqrt{1 + (y')^2}$$

$$\Rightarrow y' = \sqrt{\frac{1}{C^2} - 1} = m$$

Integrating both sides yields the general equation for $y$:

$$y = mx + c$$

Therefore, the shortest path between 2 points is a straight line.

4 Brachistochrone Curve

Another famous problem that can be solved using the Euler-Lagrange equation is the brachistochrone problem. The problem is to try to find the path that a particle would roll down the fastest.
To solve this problem a suitable lagrangian must be chosen. The time needed to travel along an infinitesimal line segment is equal to the length of the line segment divided by the speed of the particle. Let \( y(x) \) represent the vertical displacement (with the positive direction being downwards) of the particle at horizontal displacement \( x \) (it is easy to show that we needn’t worry about paths folding back on themselves), and let \( m \) be the mass of the particle.

Using conservation of energy we can find the speed the particle will be travelling at point \((x, y(x)):\)

\[
\frac{1}{2}mv^2 = mgy
\]

\[\Rightarrow v = \sqrt{2gy}\]

The lagrangian for this problem can be obtained by dividing the previous lagrangian for computing distance by the speed of the particle:

\[
L = \sqrt{\frac{1 + (y')^2}{2gy}}
\]

Since \( 2g \) is a constant we can ignore it and consider the following lagrangian:

\[
L = \sqrt{\frac{1 + (y')^2}{y}}
\]

Since \( L \) does not explicitly depend on \( x \) we can once again use the simplified form of the Euler-Lagrange equation:

\[
\sqrt{\frac{1 + (y')^2}{y}} - y \cdot \frac{\partial}{\partial y} \sqrt{\frac{1 + (y')^2}{y}} = K
\]

\[\Rightarrow \sqrt{\frac{1 + (y')^2}{y}} = \frac{(y')^2}{\sqrt{y(1 + (y')^2)}} = K
\]

\[\Rightarrow y(1 + (y')^2) = \frac{1}{K^2} = A\]

This is a separable differential equation and its solution can be given parametrically by:

\[
\begin{cases}
  x = \frac{A}{2}(\theta - \sin(\theta)) \\
  y = \frac{A}{2}(\theta - \sin(\theta))
\end{cases}
\]

This is the general form of the brachistochrone curve.
5 Catenary Curve

The catenary curve is the shape of a cable hanging under its own weight. Every physical system tends towards the state with least potential energy. In the case of a hanging cable the system will tend towards the state with the least Gravitational Potential Energy. Let \( y(x) \) be the vertical height of the section of rope with horizontal displacement \( x \) (once again we will be assuming that the rope doesn’t fold back on itself). The lagrangian of this system will be the G.P.E. associated with an infinitesimal section of rope. This quantity will be equal to the height of the section times its length times the linear mass density \( \rho \) times \( g \). Thus, the lagrangian will be the \( yg\rho \) times the lagrangian for computing distance. Like before we can ignore the constant \( g\rho \).

\[
L = y\sqrt{1 + (y')^2}
\]

There is no explicit \( x \) dependence so we can use the simplified Euler-Lagrange equation:

\[
y\sqrt{1 + (y')^2} - y' \frac{\partial}{\partial y}(y\sqrt{1 + (y')^2}) = K
\]

\[
\Rightarrow y\sqrt{1 + (y')^2} - \frac{y(y')^2}{\sqrt{1 + (y')^2}}
\]

Multiplying through by \( \sqrt{1 + (y')^2} \) gives:

\[
y = K\sqrt{1 + (y')^2}
\]

This is a separable differential equation which when rearranged gives the following integral:

\[
x = \int \sqrt{\frac{a^2}{y^2 - a^2}} dy
\]
This is a well known integral identity which gives:

\[ x = a \text{ arccosh} \left( \frac{x}{a} \right) \]

\[ \Rightarrow y = a \cosh \left( \frac{x}{a} \right) \]

Thus, the shape of a hanging cable is a cosh curve.

6 Final Comments

Hopefully, I have successfully conveyed how powerful and useful the Euler-Lagrange equation is through these 3 examples. However, it is not limited in anyway to these sorts of problems. For example, the Euler-Lagrange equation and Lagrangian mechanics forms the framework for much of theoretical physics. The principle of least action is a physical principle that describes the world around us. It states that nature always follows the path of least action (here action is the term used to describe a given functional in physics) and this principle allows us to compute the equations of motion of any system. The behaviour of a system will depend on what action you are using which is why when new physical theories are suggested it is usually given in the form of a new Lagrangian.
Since electromagnetism is a fundamentally relativistic phenomenon (relies on special relativity), the equations that govern this theory can be obtained by applying the Euler-Lagrange equation to a relativistic lagrangian. In addition, our best working theory of gravity is also formulated in terms of an action. According to the general theory of relativity, energy bends space-time, the fabric that makes up our universe. The Einstein-Hilbert action takes into account this curvature. Applying the Euler-Lagrange equation to this new lagrangian results in the Einstein Field Equations. These equations predict the bending of an object’s trajectory in the presence of a gravitational field. This is analogous to the fact that on a curved surface like a balloon the shortest path between 2 points is not a straight line but a curve and according to the principle of least action an object must follow this curved path. The Euler-Lagrange equation is also used in modelling many dynamical systems such as the dispersion of air pollutants in the atmosphere, the movement of different bodies of water in the ocean and the optimisation of rocket and submarine paths.

7 Appendix

A proof of the Euler-Lagrange equation is provided here.

Given a stationary function $y(x)$ of the functional $S$ given by:

$$S = \int_a^b L(x, y(x), y'(x)) dx$$

then any perturbation to $y$ will either increase or decrease $S$ depending on whether $y$ is a minimum or a maximum.

Let $y_\epsilon(x) = y(x) + \epsilon h(x)$ where $y_\epsilon(x)$ is a small perturbation to $y$, $h$ is a differentiable function satisfying $h(a) = h(b) = 0$ (any perturbation must preserve the start and end points) and $\epsilon$ is a small real constant. Now define $L_\epsilon$, $S_\epsilon$ to be the lagrangian and functional respectively when $y(x)$ is replaced with $y_\epsilon(x)$:

$$S_\epsilon = \int_a^b L(x, y_\epsilon(x), y_\epsilon'(x)) dx$$

Since $S_\epsilon$ is maximised/minimised when $\epsilon = 0$ by definition of $y$ it is clear that:

$$\left. \frac{dS_\epsilon}{d\epsilon} \right|_{\epsilon=0} = 0$$

Now we explicitly compute $\frac{dS_\epsilon}{d\epsilon}$:

$$\frac{dS_\epsilon}{d\epsilon} = \frac{d}{d\epsilon} \int_a^b L_\epsilon(x) dx$$
\[ = \int_a^b \frac{dL}{d\epsilon} \, dx \]

\( L, y \) and \( h \) are assumed to be sufficiently nice such that the swapping of the integral and the derivative is justified.

The next step is to compute \( \frac{dL}{d\epsilon} \). This is done by taking the total derivative:

\[
\frac{dL}{d\epsilon} = \frac{\partial L}{\partial \epsilon} \frac{d\epsilon}{dx} + \frac{\partial L}{\partial y} \frac{dy}{dx} \frac{d\epsilon}{dx} + \frac{\partial L}{\partial y'} \frac{dy'}{dx} \frac{d\epsilon}{dx}
\]

\[
= \frac{\partial L}{\partial \epsilon} \frac{dy}{dx} + \frac{\partial L}{\partial y} \frac{dy}{dx} + \frac{\partial L}{\partial y'} \frac{dy'}{dx}
\]

\[
= h(x) \frac{\partial L}{\partial \epsilon} + h'(x) \frac{\partial L}{\partial y'}
\]

where we have used the fact that \( x \) is independent of \( \epsilon \) and that \( y_\epsilon(x) = y(x) + \epsilon h(x) \).

Thus,

\[
\frac{dS}{d\epsilon} = \int_a^b \frac{dL}{d\epsilon} \, dx = \int_a^b h(x) \frac{\partial L}{\partial \epsilon} + h'(x) \frac{\partial L}{\partial y'} \, dx
\]

Substituting in \( \epsilon = 0 \):

\[
\frac{dS}{d\epsilon} |_{\epsilon=0} = \int_a^b h(x) \frac{\partial L}{\partial \epsilon} + h'(x) \frac{\partial L}{\partial y'} \, dx = 0
\]

Using integration by parts on the second term in the integral and the fact that \( h(a) = h(b) = 0 \) the equation reduces to:

\[
\int_a^b \left( \frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} \right) h(x) \, dx = 0
\]

\[
\Rightarrow \frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} = 0
\]

where the last step can be justified using the fundamental lemma of calculus of variations. This yields the Euler-Lagrange equation. The stationary function \( y(x) \) must satisfy the equation:

\[
\frac{\partial L}{\partial y} = \frac{d}{dx} \frac{\partial L}{\partial y'}
\]

If \( L \) does not explicitly depend on \( x \) then a useful simplification can be made using the fact that:

\[
\frac{dL}{dx} = \frac{\partial L}{\partial y'} \frac{d}{dx} y' + \frac{\partial L}{\partial y} \frac{d}{dx} y
\]

\[
\Rightarrow \frac{\partial L}{\partial y'} \frac{d}{dx} y' = \frac{dL}{dx} - \frac{\partial L}{\partial y} \frac{d}{dx} y
\]

8
From the Euler-Lagrange equation we have:

\[
\frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} = 0
\]

\[
\Rightarrow \frac{\partial L}{\partial y} y' - y \frac{d}{dx} \frac{\partial L}{\partial y'} = 0
\]

Assuming that \( L \) has no explicit dependence on \( x \) we have:

\[
\frac{dL}{dx} - \frac{\partial L}{\partial y} y'' - y \frac{d}{dx} \frac{\partial L}{\partial y'} = 0
\]

\[
\Rightarrow \frac{d}{dx} (L - y \frac{\partial L}{\partial y'}) = 0
\]

\[
\Rightarrow L - y \frac{\partial L}{\partial y} = K
\]

where \( K \) is an arbitrary constant.