An Integral Part of Music

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1 Introduction

The simplest of all rational functions has numerous interesting applications. I was fascinated to know that both the harmonic series (following a musical overtone series) and an imaginary horn (Gabriel's horn/Torricelli's trumpet) heavily rely on the function;

$$y = \frac{1}{x}$$

Specifically, I wanted to explore the relation between these two seemingly unrelated things, through a musical (with the help of physics) perspective.

It seemed interesting to me that such a simple function, can be manipulated with the help of a few additional concepts, to create an intriguing link. As the title states, this exploration will involve integration and musical concepts.

2 Harmonic Series

The harmonic series is a fundamental concept in music, one which most musicians may be musically aware of, but also one which most people overlook the maths of. First and foremost, the harmonic series is a divergent series, and this is important to consider given that there aren't infinite notes in a musical scale that us humans can hear. As a music lover all these years, I was surprised to learn that when a note is played, for example A (universal tuning note because middle A has a frequency of exactly 440 Hz), an overtone series commencing on A is produced. This means that what one hears is actually an infinite sum of notes, and in the case of A, an overtone series. This overtone series is closely related to the harmonic series, its just overtone series is what its referred to by musicians and harmonic series is whats its referred to by mathematicians. The A1 overtone series starts on a frequency of exactly 55 Hz. As the harmonic series is an additive series, the next note should be A2 with a frequency of 55+55=110 Hz, the following note would have a frequency of 55+55+55=165 Hz. Thus, we can refer to harmonic numbers as essentially partial sums of the harmonic series. The ratio of these higher frequencies to the base frequency decreases in size until all 12 notes of the chromatic scale are covered. For example, the ratio of 110:55 simplifies to 2:1 but the ratio of 165:110 simplifies to 3:2 and so forth in a pattern that is identical to n+1:n. Furthermore, Pythagoras upon plucking or playing (hitting a key on the piano is technically hitting a string) a string with a given length, the sound wave produced is essentially the summation of the original wavelength, half of the wavelength, a third of the wavelength and so on (technically until infinity, however the amplitude (musical volume) given at these points on a wave are negligible, which is why we don't hear dissonance when hearing one note being played). This is also why sometimes, playing a really low note gives a richer sound, because more overtones are being heard! Thus, the sum of the frequencies produced when playing an A1 can be represented as the 12th harmonic number;

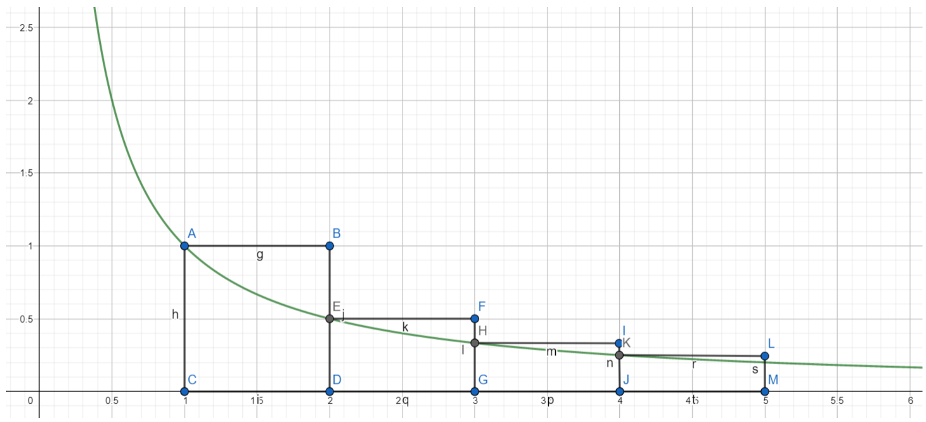
$$\sum_{k=1}^{12} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12}$$

Note that it doesn't matter what our initial base frequency is, because each subsequent frequency is added by the same amount. This is essentially a partial sum of the harmonic series, which is often generally represented as;

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

An interesting link between the harmonic and overtone series is that, the simpler the ratios, the more 'harmonically consonant' the notes are. For example, ratio of 1:2 represents an octave, which is an interval of eight, comparing start to finish. A ratio of 2:3 represents a perfect fifth above the octave, this may also explain why perfect fifths are often used in music (as they are the simplest distinct ratio of notes). A ratio of 3:4 represents a perfect fourth and so on. Moving forwards, instead of graphing the oscillating wave function that is produced when playing a certain frequency, I noticed that the summation element is essentially the reciprocal function I mentioned earlier, that is: f(x)=1/x. Doing some research, I found that the harmonic series is indeed related to this function, and the link is not only ingenious, but is beneficial for the further analysis that is yet to come.

Restricting the domain of y=1/x from x=1 to infinity, notice that upon constructing rectangles with width 1 and upper left corner touching the graph y=1/x we are left with rectangles which have heights that line up with terms within the harmonic series. A diagrammatic representation of the first few rectangles are shown in the image below (constructed using GeoGebra by myself)



Notice that since all their widths are 1 unit, the sum of all their areas is congruent to the harmonic series. However, integration tells us the area under the curve given our bounds;

$$\int_1^n \frac{1}{x} = \ln(n)$$

In fact, the natural logarithm is often used as an approximation for the harmonic series, and even more interestingly, the sum of the area difference (between the rectangles and integral) actually tends towards a constant, the Euler-Mascheroni constant which is denoted by small gamma.

$$H_n \approx \ln(n) + \gamma$$

The fact that we started with frequency multiples and scales and ended with integrals and logarithms already says a lot about the link between math and music, and even more about the reason why conventional music is so 'natural'.

3 Gabriel's Horn

Gabriel's horn, also known as Torricelli's Trumpet is widely considered as an impossible object, despite its precise mathematical definition and proof. It is essentially the object obtained from rotating the curve y=1/x about the x-axis. Doing this requires volume of revolution;

$$V = \pi \int_1^a \frac{1}{x^2}\, dx$$

and since a approaches infinity, as the original curve does,

$$V = \lim_{a \to \infty} \pi(\frac{1}{a} - 1)$$

$$V = \pi$$

Surface integration of a solid of revolution can be applied to show that in fact, Gabriel's horn has infinite surface area! While this is true, I won't be using this property moving forwards. This means that the horn (an impossible instrument) has a finite volume and an infinite surface area. This gave rise to the painter's paradox, as you can fill the horn with a fixed volume of paint but can never paint the entire surface of the horn. However, we are going to look at the link between the two things that we have discovered, that is, the harmonic series and Gabriel's horn.

4 A Bit of Music and Physics

By the same principles of the painter's paradox, sound could be produced from the horn, despite it being impossible to even hold in someone's hands. It is tough to even visualise this, but given that the horn has an exact and definite theoretical volume, we can actually compute the nature of the sound produced by this horn! The first equation, derived from solving the Restating Euler-Lagrange differential equation, yields;

$$f_c = \frac{mv}{4pi}$$

where f is the cut-off frequency, v is the speed of sound and m is a lesser-known 'flare constant' that we can compute for. In order to do this, we need another equation relating our known variables found earlier;

$$S_l = \frac{v}{(2f_c)^2\pi}$$

where S is the cross-sectional area of the horn. Since we are using Gabriel's horn, the cross-sectional area is equivalent to the area of a circle, as the horn is obtained from rotating the graph y=1/x. The area is simply;

$$S_l = \pi r^2$$

where r is 1,

$$S_l = \pi$$

Yet again π reappears, and now we know that the cross-sectional area is equal to the volume of Gabriel's horn. Now we can calculate the cut-off frequency for the horn by rearranging the equation (assuming the speed of sound is 343 m/s);

$$\pi = \frac{343}{(2f_c)^2\pi}$$

$$4(f_c)^2 = \frac{343}{\pi^2}$$

$$f_c = \sqrt{\frac{343}{4\pi^2}}$$

Note that the cut-off frequency value must be non-negative, thus;

$$f_c \approx 2.948\,\text{Hz}$$

Comparing this frequency value to real life, we cannot hear this frequency, as the human hearing range is 20Hz-20000Hz. But as we know from earlier, the sound produced can be defined using the overtone series. This is helpful as we can pinpoint what note will be played, the frequency at which it is played, and even the volume at which it is played. The lowest audible note is an E0, with a frequency of 20.601Hz. Given that our base frequency is 2.948Hz, we can perform repeated addition to determine the nature of the tonic (first note in scale).

$$2.948\text{Hz} \times 7 = 20.636\text{Hz}$$

Note that this frequency value is very close to E0, with a difference of 0.035Hz. Given that the value of the gamma constant is 0.57721, it could be reasonably assumed that a note with this frequency is virtually identical to E0 (to the human ear). After all, no piano is ever exactly in tune (since each note is multiplied by a twelfth root of 2 from the base frequency). Given that we had to multiply our base frequency 7 times to reach E0, we have to work in reverse to find out our tonic note. Specifically, E0 is the supermajor second above the subminor third above the minor third above the major third above the perfect fourth above the perfect fifth above the octave of our base frequency. Fortunately for us, we haven't entered the realm of microtones, and in fact, the previous relation between E0 to our base note simplifies to, E0 being 3 octaves above our tonic, which means that the Gabriel's horn would theoretically produce a raw sound equivalent to the frequency of a E(-3) (and thus the overtone series commencing on this note). Obviously, this is not possible, but it's interesting to see that we have performed mathematical calculations on an impossible object, using very real and possible parameters to obtain yet again, another impossible (at least to hear) result.

5 Conclusion

Although the Gabriel's horn as an object, is impossible, I have found the theoretical frequency of the sound that would be produced, if someone managed to play it, which as we know, is also impossible. The harmonic series helped us understand the overtone series, and thus enabled us to deduce the tonic note produced given the relation to known frequencies (E0). Knowing the intervals between each note, I was lucky that the relation was three times a perfect octave. However, I have just found the frequency of the note, and I am sure that it would be possible to determine the amplitude and wave function, perhaps by applying Fourier series, to deduce the waveform produced by Gabriels' horn. Waveforms of instruments help us better understand how different musical instruments produce different sounds, and this information is implemented in sound design. All in all, this exploration was just based on my curiosity on analysing an impossible yet very probable instrument. I think that an integral part of music is understanding that any form or manner of sound can be interpreted as musically beautiful, given the wonderful relationship between music and math.