Can you cut a square pizza into equal triangular slices among an odd number of people?

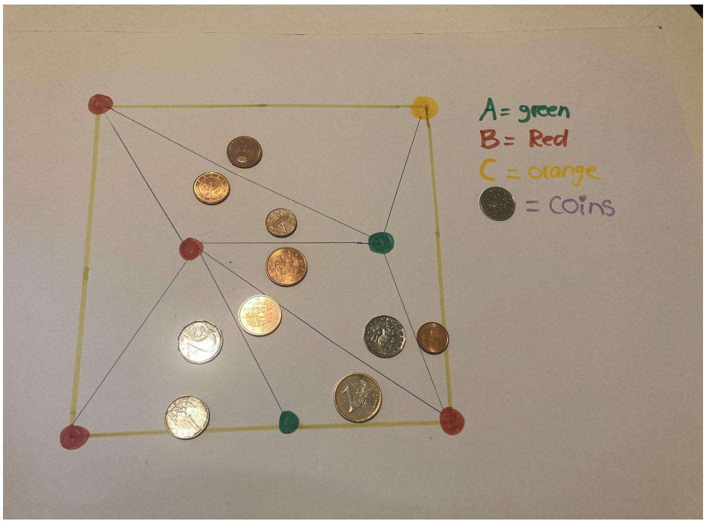
After reading the prompt for this essay; which is to write a short article about your favourite mathematical topic, the memories of trying to cut my mum's famous square panned pizza into 7 equal parts (one for each family member) immediately sprang to mind. The awkwardness of having to give one unlucky family member a smaller slice than the rest, is always a recurring scenario; no matter how hard I try to equally cut the slices.

So, why can I never cut my mum's square pizza into equal sized triangular slices for our family of 7?

I introduce to you: Sperner's lemma and 2-adic valuations

Sperner's Lemma

Sperner's lemma says that if we triangulate a square (cut it into triangles) and label every verticy A,B or C then that means that the number of AB edges on the perimeter (has a vertex that's coloured green for A and another coloured red for B) and the number of ABC triangles (has a vertex coloured green,red and orange for A,B,and C) have the same parity (congruent modulo of 2)



AB edges on perimeter=ABC triangles modulo 2

To prove Sperner's theorem:
Place a coin on each side of the AB edges and count those coins.
When counting use these 2 methods:
-Every ABC triangle gives you one coin (from the AB edge)
-Any other triangles give you 0 or 2 coins.
With either method, the number of coins is congruent modulo 2 to the number of ABC triangles.

Similarly, each AB edge on the perimeter gives you one 1 coin but other edges give you 0 or 2 coins. Consequently, the number of coins is congruent modulo 2 to the number of AB edges on the perimeter. This means that the number of AB edges=the number of ABC triangles!

In conclusion, regardless of how you chose to cut your square into triangles or colour the vertices, the number of AB edges on the perimeter and the number of complete triangles (ABC triangles) are always congruent to modulo 2.
To prove that there's an odd number of ABC edges on the perimeter, we can prove the existence of an ABC triangle .

2-adic valuations

Valuations are functions that measure how big a number is, 2-adic valuation is simply just another way to measure the size of a rational number.

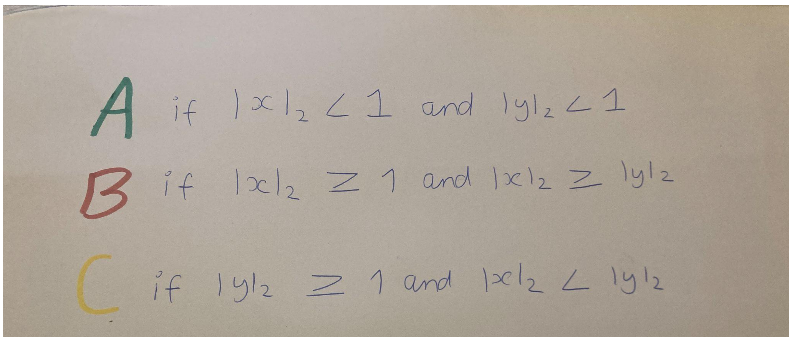
For instance the rational number x=y/z
Can be written as $x=2^{n} \cdot y/z$ (for odd numbers y and z)
The 2-adic valuation of this is $|x|_2=(1/2)^{n}$

Here's a few example of this method in use $\rightarrow |4|_2 = 1/4$
$\rightarrow |6|_2 = 1/2$
$\rightarrow |0|_2 = 0$

It's important to note the triangular equality (the size of x+y is not bigger than the size of x + the size of y),because in 2-adic valuations this concept is actually
$|x+y|_2 \leq \text{maximum}\{|x|_2, |y|_2\}$

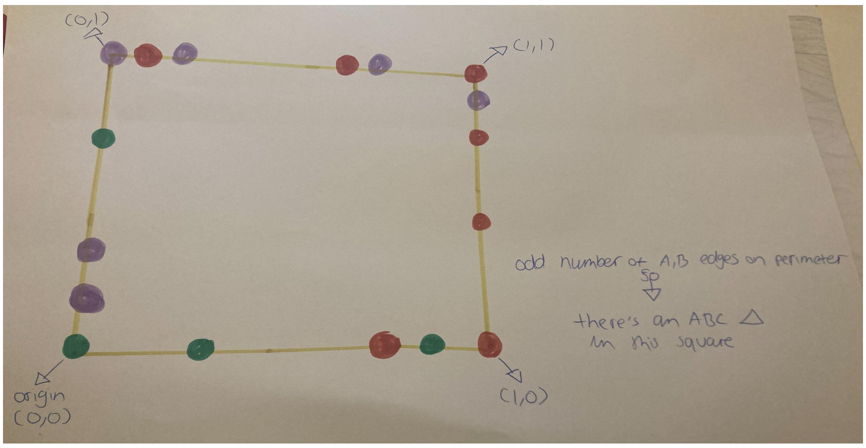
If $|x|_2 \neq |y|_2$ then $|x+y|_2 = \text{maximum}\{|x|_2, |y|_2\}$

Applying the 2 adic valuations and Sperner's theorem to the question



Firstly, implement the 2-adic valuation to help decide which vertices will have which colour.

Sperner's lemma suggests that there is an ABC triangle somewhere in this square. That triangle will have an area of 1/n since the area of the whole square is 1, if r is the area of an ABC triangle then $|r|_2 \geq 2$.



If we say that n is the number of equal area triangles and they each have an area of r=1/n, then

$|n \cdot r|_2 = |1|_2 = 1.$

On the condition that we colour the vertices in the triangle, apply Sperner's lemma to get an ABC triangle and then implement the calculation for the 2-adic valuation of the area r, we see that $|r|_2 \geq 2$ which means that $|n|_2 \leq \frac{1}{2}$. This is less than 1 so n can be divided by 2, making n even!

Thus, if you split a square into n triangles with each triangle having an area of 1/n then n is always even!

-Kenzie Salem