Complete Quadrilaterals

Helia Kalanaki

Contents

1 Introduction

1.1 Preliminaries

Glossary

cline A cline (or generalized circle) refers to either a circle or a line.. 2

concurrent Two or more lines are said to be concurrent if they intersect in a single point.[8]. 4

disjoint Two sets are said to be disjoint if there are no common elements. In other words, when the intersection of the two sets is empty, then those sets are said to be disjoint sets.[1]. 10

orthocenter The point of concurrence of the altitudes of a triangle is called the orthocenter of the triangle.. 6, 13

Definition 1.1 (Power of a Point) We define the power of P with respect to the circle ω by

$$Pow_{\omega}(P) = OP^2 - r^2.$$

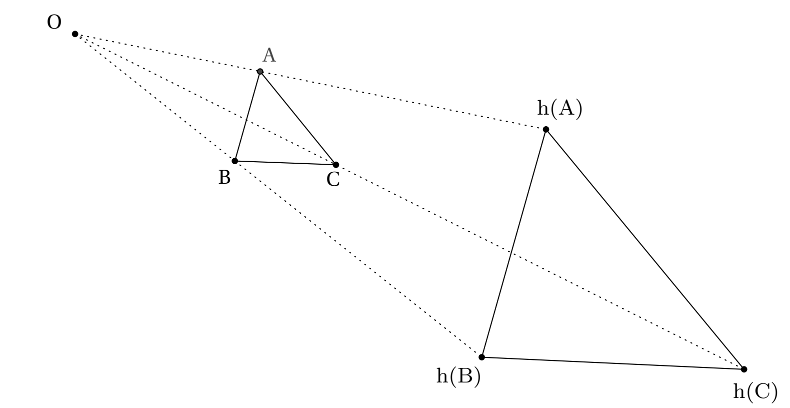
This quantity may be negative. Actually, the sign allows us to detect whether P lies inside the circle or not. [4]

Definition 1.2 (radical axis) The radical axis of two circles is a line that is the locus of all points that have equal powers with respect to both circles. [6]

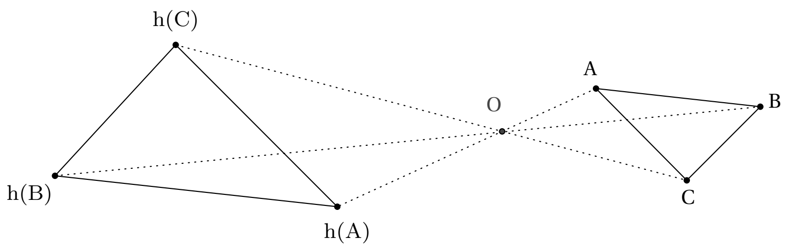
Definition 1.3 (coaxial) If a set of circles have the same radical axes, then we say they are coaxial. [4]

Definition 1.4 (Homothety) A homothety h is a transformation defined by a center O and a real number k. It sends a point P to another point $h(P)$, multiplying the distance from O by k. The number k is the scale factor. It is important to note that k can be negative, in which case we have a negative homothety. [4]

A homothety h with center O acting on $\triangle ABC$:

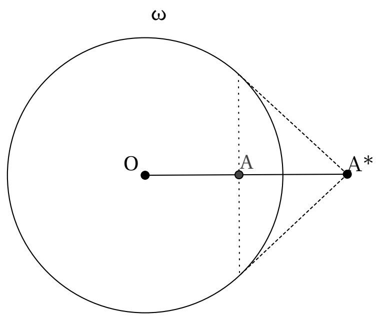


A negative homothety with center O:



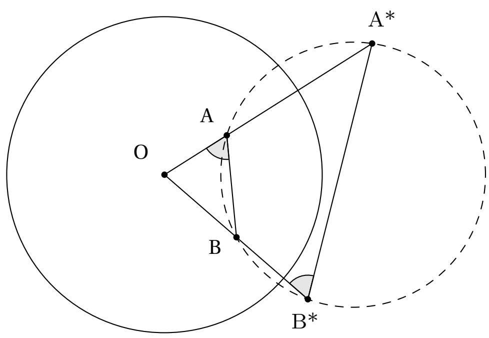
Definition 1.5 (Inversion) Inversion is useful for turning circles into lines and for handling tangent figures. The idea is to view every line as a circle with infinite radius. We add a special point P_∞ to the plane, which every ordinary line passes through (and no circle passes through). This is called the point at infinity. Therefore, every choice of three distinct points determines a unique cline—three ordinary points determine a circle, while two ordinary points plus the point at infinity determine a line. With this said, we can now define an inversion. Let ω be a circle with center O and radius R. We say an inversion about ω is a map (that is, a transformation) which does the following.
-The center O of the circle is sent to P_∞.
-The point P_∞ is sent to O.
-For any other point A, we send A to the point A^* lying on ray OA such that $OA \times OA^* = r^2$.
Try to apply the third rule to $A = O$ and $A = P_\infty$, and the motivation for the first two rules becomes much clearer. The way to remember it is "$\frac{r^2}{0} = \infty$" and "$\frac{r^2}{\infty} = 0$".
At first, this rule seems arbitrary and contrived. What good could it do? First, we make a few simple observations.

1. A point A lies on ω if and only if A = A*. In other words, the points of ω are fixed.
2. Inversion swaps pairs of points. In other words, the inverse of A* is A itself. In still other words, $(A^*)^* = A$. [4]



Lemma 1.6 (Inversion and Tangents) Let A be a point inside ω, other than O, and A* be its inverse. Then the tangents from A* to ω are collinear with A.

This configuration is shown in the figure above. We just need to check that $OA.OA^* = r^2$. Inversion is not very interesting if we only look at one point at a time
-how about two points A and B?

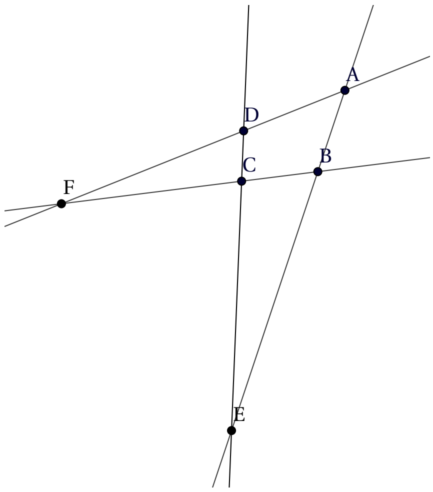


This situation is shown in the figure above. Now we have some more structure. Because $OA.OA^* = OB.OB^* = r^2$, by power of a point we see that quadrilateral ABB*A* is cyclic. Hence we obtain the following theorem.

Theorem 1.7 (Inversion and Angles) If A* and B* are the inverses of A and B under inversion centered at O, then $\angle OAB = -\angle OB^*A^*$.

2 Complete Quadrilaterals

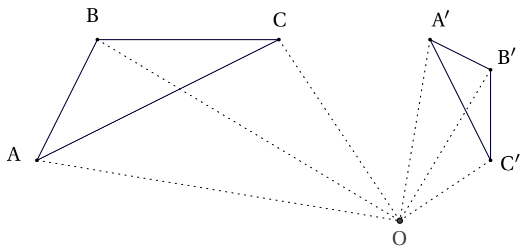
Definition 2.1 (Complete Quadrilateral) A complete quadrilateral is the figure determined by four lines no three of which are concurrent. The most common configuration in which a quadrilateral ABCD and one takes the intersections $E = AB \cap CD$ and $F = DA \cap BC$. [2] In addition, the points could be split into three pairs such that the connecting segments do not belong to any of the given lines. These three segments are called diagonals of the quadrilateral. [3] [5]



For instance the diagram above is the complete quadrilateral ABCDEF has three diagonals: AC, BD, and EF.(Because, according to the definition above, these segments are the only ones which do not belong to any of the lines AD, BC, AB,and DC.)

2.1 Spiral Similarity

Definition 2.2 (Spiral Similarity) A spiral similarity centered at a point O is a composition of a dilation and a rotation with respect to O. [2]



The most commonly occurring case of a spiral similarity is between two segments. Consider a spiral similarity at O mapping a segment $\overline{AB}$ to $\overline{CD}$. [4]

$\triangle$OAB is similar to $\triangle$OCD. We now determine O in terms of A, B, C, D via complex numbers. It is easy to check that

$$\frac{c-o}{a-o} = \frac{d-o}{b-o}.$$

That implies

$$o = \frac{ad - bc}{a + d - b - c}.$$

So O is uniquely determined by A, B, C, D. That implies in general there is exactly one spiral similarity taking any segment to any other segment. The exception is if ABDC is a parallelogram, since then $a + d = b + c$ and the spiral similarity fails to exist.

Lemma 2.3 (Spiral Centers) Let $\overline{AB}$ and $\overline{CD}$ be segments, and suppose $X = \overline{AC} \cap \overline{BD}$. If (ABX) and (CDX) intersect again at O, then O is the center of the unique spiral similarity taking $\overline{AB}$ into $\overline{CD}$. [4]

Proof This is actually just a matter of angle chasing. We have

$$\angle OAB = \angle OXB = \angle OXD = \angle OCD$$

and similarly

$$\angle OBA = \angle ODC.$$

That implies $\triangle OAB \sim \triangle OCD$, which is sufficient. ■

Whenever all six points in the figure appear, we automatically have a pair of similar triangles. By now, an observant reader may have realized that there is more than one set of similar triangles in the figure. We see that in fact, $\triangle OAC \sim \triangle OBD$ as well. After all, $\angle AOC = \angle BOD$ and $\frac{AO}{CO} = \frac{BO}{DO}$ (the ratios arising from the original spiral similarity). What this means is that spiral similarities occur in pairs. More precisely, we get the following proposition.

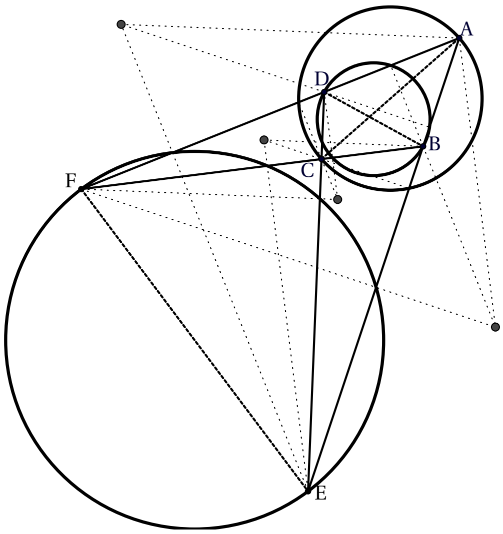
Lemma 2.4 The center of the spiral similarity taking $\overline{AB}$ to $\overline{CD}$ is also the center of the spiral similarity taking $\overline{AB}$ to $\overline{CD}$. [4]

Thus we have a second spiral similarity, but this time we know its center. What happens if the Spiral Centers Lemma is applied again, this time in the other direction? Does this really mean that $AB \cap CD$ lies on (AOC) and (BOD) as well? That is precisely Miquel's theorem.

2.2 Newton-Gauss Line

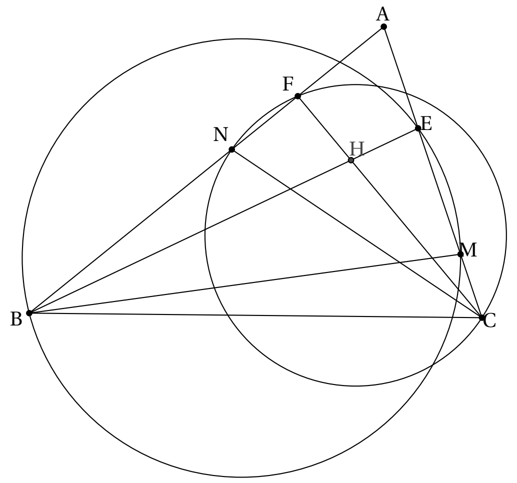
Definition 2.5 (Newton-Gauss Line) Consider the three diagonals of a complete quadrilateral, namely $\overline{AC}$, $\overline{BD}$, $\overline{PQ}$. It turns out their midpoints are collinear. The line through them is called the Gauss line (sometimes also called the Newton-Gauss line). [4]

Theorem 2.6 (Gauss-Bodenmiller Theorem) The circles with diameters $\overline{AC}$, $\overline{BD}$, $\overline{FQ}$ are coaxial. Their radical axis is a line passing through each of the four orthocenters of the triangles $\triangle FAB$, $\triangle FCD$, $\triangle EAD$, $\triangle EBC$. The radical axis is sometimes called the Steiner line (or sometimes Aubert line). [7]



Proof We begin with the following lemma:

Lemma 2.7 Let $\triangle ABC$ be a triangle and let M, N be points on sides CA and AB respectively.Then the orthocenter H of triangle $\triangle ABC$ lies on the radical axis of the circles with diameters BM and CN. [7]



Proof Let E, F be the feet of the B, C-altitudes in triangle ABC respectively. Then $\angle$BEM $= \angle$CEF $= 90°$ so E lies on the circle with diameter BM and F lies on the circle with diameter CN. Hence, it suffices to show that HB.HC = HC.HF (so that the power of H with respect to the two circles is equal). But since the reflection of H over the sidelines of the triangle $\triangle$ABC lie of the circumcircle of triangle $\triangle$ABC we have that HB.HE and HC.HF are both equal to half the power of H with respect to the circumcircle of triangle $\triangle$ABC. This completes the proof of the lemma. ■

Let H_1,H_2,H_3,H_4 be the orthocentres of triangles $\triangle$ABF, $\triangle$BCE, $\triangle$CDF, $\triangle$DAE respectively. Returning to the problem, note that segments $\overline{AC}$, $\overline{BD}$, and $\overline{FE}$ are cevians in triangle $\triangle$ABF so from the claim we know that H_1 the radical centre of the circles wit diameters $\overline{AC}$, $\overline{BD}$, $\overline{EF}$. Similarly, points H_2, H_3, H_4 are also radical centres of these circles. Hence, either these circles are coaxial or the orthocenters of triangles $\triangle$ABF, $\triangle$BCE, $\triangle$CDF, $\triangle$DAE coincide. But the latter situation is clearly impossible, so this completes the proof. ■

So that we can conclude the following:

Theorem 2.8 (newton-line-thm) Let ABCD be a quadrilateral and let E = AB $\cap$ CD and F = DA $\cap$ BC. Then the midpoints of the segments $\overline{AC}$, $\overline{BD}$, $\overline{EF}$ are collinear. [7]

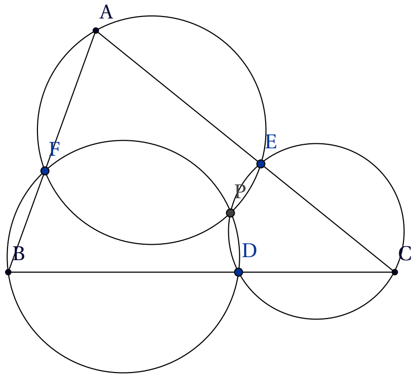
2.3 Miquel Point

Theorem 2.9 (Miquel's Pivot Theorem) Let ABC be a triangle and let D,E,F be points lying on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively. Then the circumcirles of triangles $\triangle$AEF, $\triangle$BFD, $\triangle$CDE concur. [7]

Proof Let the circumcircles of the triangles $\triangle BDF$ and $\triangle CDF$ intersect again at P. Then we have that:

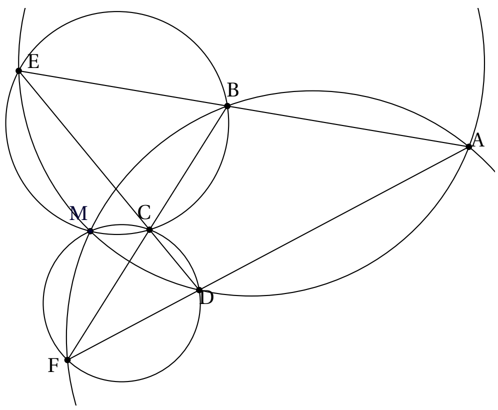
$$\angle EPF = 360^\circ - \angle FPD - \angle DPE = 360^\circ - (180^\circ - \angle B) - (180^\circ - \angle C) = 180^\circ - \angle A$$

So quadrilateral $AEPF$ is cyclic. This completes the proof. ■

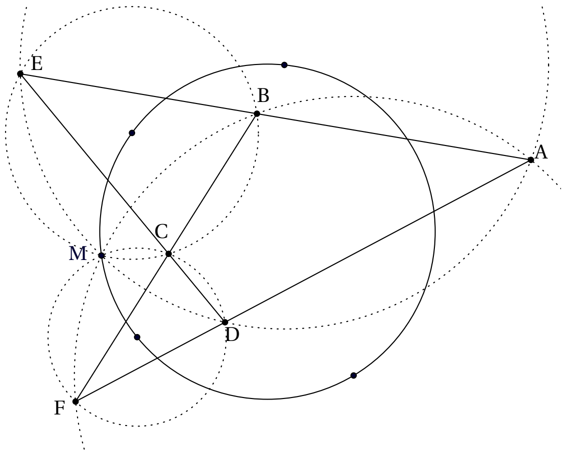


Theorem 2.10 (Miquel's Theorem) Let $ABCD$ be a quadrilteral and let $E = AB \cap CD$ and $F = DA \cap BC$. Then the circumcircles of triangles $\triangle ABF$, $\triangle BCE$, $\triangle CDF$, $\triangle DAE$ concur at a point M, called the Miquel Point of complete quadrilateral $ABCDEF$. [7]

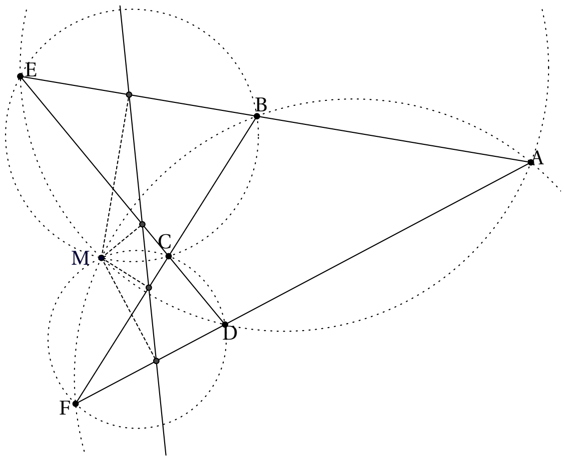
Proof Applying Miquel's Pivot Theorem to triangle $\triangle ABF$ with points C, D, E we have that the circumcircles of triangles $\triangle DAE$, $\triangle BCE$, and $\triangle CDF$ concur. Applying Miquel's pivot Theorem three more times in the same way then yields the desired result. ■



Lemma 2.11 (Centers are Concyclic with the Miquel Point) The four centers of (PAB), (PDC), (QAD), (QBC) lie on a circle passing through the Miquel point. [4]



Lemma 2.12 (Altitudes from the Miquel Point) The feet of the perpendiculars from M to lines AB, BC, CD, DA are collinear. Furthermore, the line though these four points is perpendicular to the Gauss line. [4]



Theorem 2.13 (Miquel Point of a Cyclic Quadrilateral) Let ABCD be a cyclic quadrilateral inscribed in circle ω with diagonals meeting at R. Then the Miquel point of ABCD is the inverse of R with respect to inversion aroundω.

Proof Let O be the circumcenter of ABCD, and let R^* be the image of R. It suffices to show $R^* = M$. Angle chasing (left as an exercise) lets us establish $\angle AR^*B = \angle APB$, so that R^* lies on (PAB). Similarly, R^* lies on (PCD), (QBC), and (QDA). Hence R^* is indeed the Miquel point. ■

3 Relationship with the Projective Geometry

Definition 3.1 (Cross Ratios) The cross ratio is an important invariant in projective geometry. Given four collinear points A, B, X, Y (which may be points at infinity), we define the cross ratio as

$$(A, B; X, Y) = \frac{XA}{XB} \div \frac{YA}{YB}$$

Here the ratios are directed with the same convention as Menelaus's theorem; in particular, the cross ratio can be negative! If A, B, X, Y lie on a number line then this can be written as

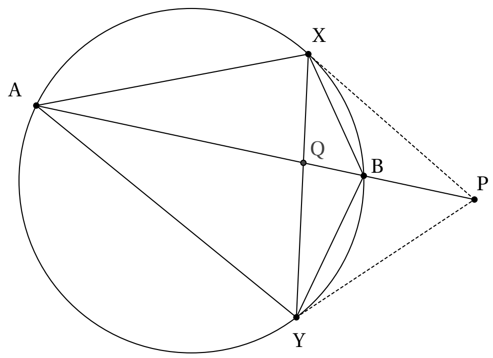
$$(A, B; X, Y) = \frac{x - a}{x - b} \div \frac{y - a}{y - b}$$

You can check that $(A, B; X, Y) > 0$ precisely when segments $\overline{AB}$ and $\overline{XY}$ are disjoint or one is contained inside the other. We also generally assume $A \neq X$, $B \neq X$, $A \neq Y$, $B \neq Y$. [4]

3.1 Harmonic Bundles

Definition 3.2 (Harmonic Bundles) The most important case of our cross ratio is when $(A, B; X, Y) = -1$. We say that $(A, B; X, Y)$ is a harmonic bundle in this case, or just harmonic. Furthermore, a cyclic quadrilateral AXBY is a harmonic quadrilateral if $(A, B; X, Y) = -1$. Observe that if $(A, B; X, Y) = -1$, then $(A, B; Y, X) = (B, A; X, Y) = -1$. We sometimes also say that Y is the harmonic conjugate of X with respect to AB; as the name suggests, it is unique, and the harmonic conjugate of Y is X itself. [4]

Lemma 3.3 ((9.9) (Harmonic Quadrilaterals)) Let ω be a circle and let P be a point outside it. Let PX and PY be tangents to ω. Take a line through P intersecting ω again at A and B. Then
(a) AXBY is a harmonic quadrilateral.
(b) If $Q = AB \cap XY$, then $(A, B; Q, P)$ is a harmonic bundle



Lemma 3.4 (Inversion Induces Harmonic Bundles) Let P be a point on line AB, and let P^* denote the image of P after inverting around the circle with diameter AB. Then $(A, B; P, P^*)$ is harmonic.

Lemma 3.5 ((9.11)Cevians Induces Harmonic Bundles) Let $\triangle ABC$ be a triangle with concurrent cevians AD, BE, CF (possibly on the extensions of the sides). Line EF meets BC at X (possibly at a point at infinity). Then $(X, D; B, C)$ is a harmonic bundle. [4]

Lemma 3.6 (Complete Quadrilaterals Induces Harmonic Bundles) Let ABCD be a quadrilateral whose diagonals meet at K. Lines AD and BC meet at L, and line KL meets AB and CD at M and N. Then $(K, L; M, N)$ is a harmonic bundle. [4]

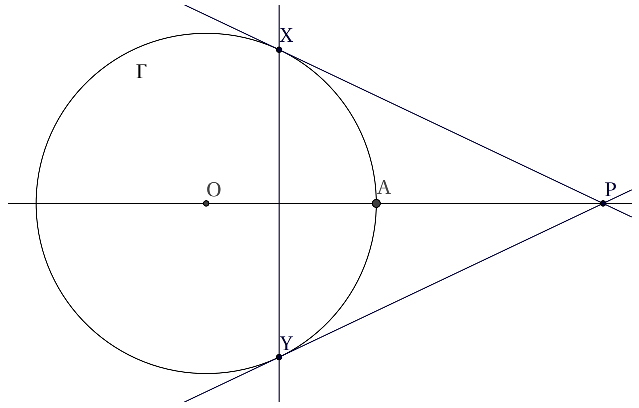
Proof Let $P = AB \cap CD$, and let $Q = PK \cap BC$. By Lemma 3.5, $(Q, L; B, C) = -1$. Projecting onto the desired line, we derive

$$-1 = (Q, L; B, C) \stackrel{P}{=} (K, L; M, N).$$

. ■

3.2 Pole and Polar

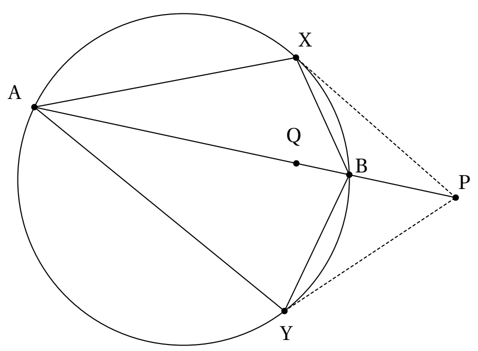
Definition 3.7 (Poles and Polars) Let the image of the point P under inversion with respect to the circle with center O and radius r be P_0. The polar of P is the line p perpendicular to the line OP at P_0. In this case, the point P is called the pole of p. [6]



Theorem 3.8 (La Hire's Theorem) A point X lies on the polar of a point Y if and only if Y lies on the polar of X. [4]

La Hire's theorem demonstrates a concept called duality: one can exchange points for lines, lines for intersections, collinearity for concurrence. Simply swap every point with its polar and every line with its pole. We can now state an important result relating poles and polars to harmonic bundles.

Lemma 3.9 Let $\overline{AB}$ be a chord of a circle ω and select points P and Q on line AB. Then $(A, B; P, Q) = -1$ if and only if P lies on the polar of Q. [4]

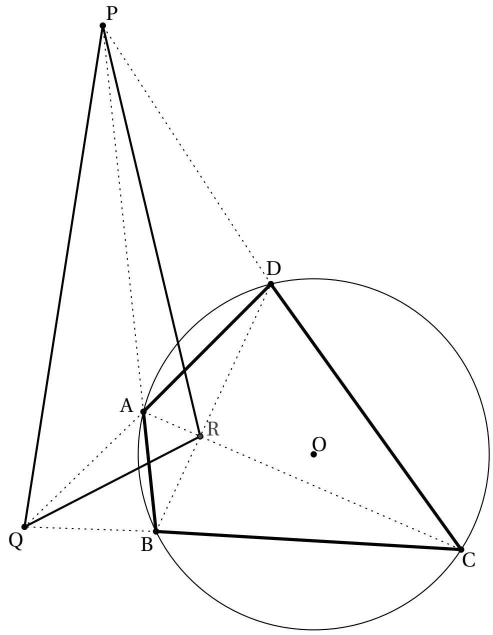


Proof We consider only the case where P is outside ω and Q is inside it. Construct the tangents PX and PY to ω. Lemma 3.3 gives $(A, B; P, XY \cap AB) = -1$, so Q lies on the polar of P (namely line XY) if and only if $(A, B; P, Q) = -1$. ■

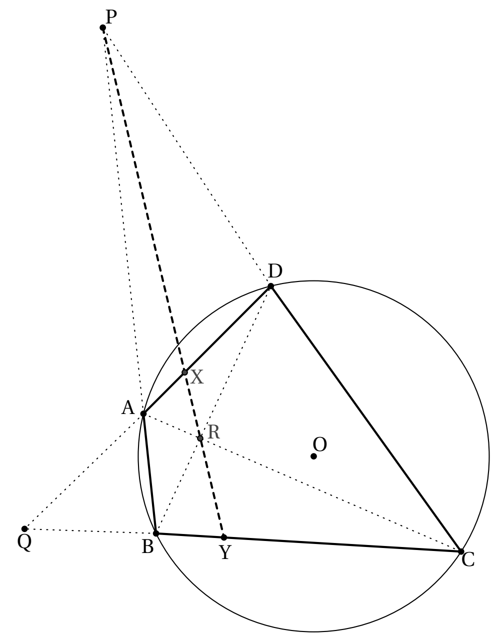
3.3 Brocard Theorem

Theorem 3.10 box:

Theorem 3.10 (Brocard's Theorem) Let $ABCD$ be an arbitrary cyclic quadrilateral inscribed in a circle with center O, and set $P = AB \cap CD$, $Q = BC \cap DA$, and $R = AC \cap BD$. Then P, Q, R are the poles of QR, RP, PQ, respectively. In particular, O is the orthocenter of triangle $\triangle PQR$. [4]



We say that triangle $\triangle PQR$ is self-polar with respect to ω , because each of its sides is the polar of the opposite vertex.



Proof First, we show that Q is the pole of line PR. Define the points $X = AD \cap PR$ and $Y = BC \cap PR$. By Lemma 3.5, both $(A, D; Q, X)$ and $(B, C; Q, Y)$ are harmonic bundles. Therefore, X and Y both lie on the polar of Q, by Lemma 3.9. Since the polar of Q is a line, it must be precisely line XY, which is the same as line PR. The same can be used to show that P is the pole of line QR and R is the pole of line PQ; projective geometry is immune to configuration issues. (This is part of the reason we like points at infinity.) This gives that PQR is indeed self-polar. Finally, the definition of a polar implies that O is the orthocenter of triangle $\triangle PQR$, completing the proof. ■

References

[1] disjoint-sets. https://www.cuemath.com/algebra/disjoint-sets/.

[2] Jafet Baca. On a special center of spiral similarity. https://www.awesomemath.org/wp-pdf-files/math-reflections/mr-2019-01/mr_1_2019_spiral_similarity.pdf, 2019.

[3] Alexander Bogomolny. Theorem of complete quadrilateral what is it? a mathematical droodle. https://www.cut-the-knot.org/Curriculum/Geometry/Quadri.shtml.

[4] Evan Chen. *Euclidean Geometry in Mathematical Olympiads*. Mathematical Association of America, 2016.

[5] J. L. Coolidge. *A Treatise On the Circle and the Sphere*. AMS - Chelsea Publishing, 1971.

[6] Stefan Lozanovski. *A Beautiful Journey Through Olympiad Geometry*. 2016.

[7] Cosmin Pohoata Titu Andreescu, Sam Korsky. *Lemmas in Olympiad Geometry*. XYZ Press, 2016.

[8] Eric W. Weisstein. Concurrent. https://mathworld.wolfram.com/Concurrent.html.