Digit Sums, and The Importance of 9 in The Modern Number System

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To begin, I'd like to include that the idea of digit sums, or digital sums as it were, are archaic in nature. They illustrate the basic concepts in arithmetic, as well as the fundamentals by which we understand our current number system, how it functions, and the patterns that only present themselves when one is on the search for beauty. With this being said, these archaic concepts fail to catch the eyes of young and aspiring mathematicians like myself, despite their importance. However, let me assure you that the patterns discovered and explored at a fundamental level are, in my humble opinion, the most efficient way to understand the language of mathematics. By improving the forgotten methods, we shall uncover more information regarding the origins of mathematics and therefore, improve our understanding of the ever-changing world around us. Take, for instance, the fact that some of Ramanujan's work found in his "lost" notebook is being employed to understand black holes. Another brief example can be found when talking about Katherin Johnson and the team of African-American women at NASA, and how they utilized Euler's method to calculate the necessary trajectory from the Earth to the Moon for US Apollo missions. As we can see, older mathematics often holds the key to modern problems because mathematics is in essence, very versatile. I decided to embark on a journey to fabricate a mathematical expression that essentially illustrates the relationships between natural numbers and their digit sums. However I should mention that this idea of "casting out nines" has been around for ages, but the equation I wrote is just a different way to express it. Now, without further delay, let us dive into the absolutely wonderful world of ancient arithmetic with a modern twist.

Digit sums have been called by many names, but the term "digit sum" and "digital sum" stuck mainly because we can mathematically express a sum as the greek letter "$\sum$"(Sigma). This allows us to fabricate what's called an equation which can be simply defined as a statement that the values of two mathematical expressions are equal. For instance, we can establish that $x+y=x+y$, or of course by the commutative property, $x+y=y+x$. Also, we can define a digit sum as essentially all of the digits present in that number in a given interval(with or without constraints) added together. So, now that some basic definitions have been stated we can go about explaining the following expression:

$$(k-1)x+\sum_{n=0}^{k-1} d_n = n; \exists \, \forall n, n \in \mathbb{N}$$

This mathematical expression has multiple parts so let us navigate each one of the elements separately.

$$\sum_{n=0}^{k-1} d_n$$

This is the most vital element of the expression and is the only part that is not a result of my own personal work. So, since it is the backbone, what does it tell us? This small, yet elegant expression accurately describes what a digit sum is all by itself. Starting from left to right we immediately see the sum symbol represented by the greek letter sigma, as explained before. We can see that there are two smaller expressions, one of which is sitting below, and the other is simply resting on top of sigma. This essentially means the sum of something from zero to k-1. Now, "n" is just a natural number which is explained later in the expression, and "k" is the number of the base we are in. Interestingly enough, "k" could theoretically be anything. However, the expression we have developed really only works if we are in base 10 as far as I can tell but of course, I haven't got the chance to explore all of the possible bases. So, these terms on the top and the bottom essentially give us an interval to stay between. Being in base 10 makes life much easier because staying within the interval from zero to 10 minus one, it saves us from encountering an absolutely horrendous combinatorial nightmare. To elaborate, I have to explain the next term first. The other element located to the right of sigma I'll say, "d sub n" just means "digits of n," which we have already determined is a natural number. So in full: The sum of the digits of some natural number "n" from zero to 9, is what we end up interpreting this as. So, if I had the number "2345," This expression tells us to add the digits given the interval from zero to nine (*i.e.* 2+3+4+5). That is why this expression, especially the interval, is crucial.

Now, this is fantastic, but it fails to tell us anything of real value with the exception of just being a cool way to symbolically express a sentence and define an interval. This is where the next term comes into play:

$$(k-1)x$$

This term is very similar to the last one, namely because of the "k-1" we saw resting upon the top of sigma. However, notice the slight difference in the structure. We can see that this seemingly important "k-1" expression which was previously defined as the base number minus one is now being multiplied by some number "x." Furthermore, we can logically conclude that whatever "x" may be, it will always be divisible by the base number minus one. In our case in base 10, this would mean that our variable will always be divisible by nine. Lastly, let me reiterate the original mathematical statement and put it into words:

$$(k-1)x+\sum_{n=0}^{k-1} d_n = n; \exists\, \forall n, n \in \mathbb{N}$$

We know now that by virtue of definition this is telling us any natural number "n" is equal to the base number minus one multiplied by some number "x" in addition to the natural numbers' digit sum given the interval from zero to the base number minus one.

Now for our typical base ten system we would be able to rewrite the expression as follows:

$$9x+\sum_{n=0}^{k-1} d_n = n$$

Let us see this expression in action because, after all, this article is all about the patterns and what it reveals about our base ten natural number system. Let us imagine we were given the number 3572. According to our equation, we get the following:

$$9x+(3+5+7+2)=3572$$
$$9x+17=3572$$
$$9x=3555$$
$$x=395$$

Well, how else can we get this number, and is it correct? Let's write the original number in a different but equivalent form:

$$(3\times 1000)+(5\times 100)+(7\times 10)+2$$

Notice how we can rearrange this as the following:

$$(3\times(999+1))+(5\times(99+1))+(7\times(9+1))+2$$
$$(3\times 999)+(5\times 99)+(7\times 9)+3+5+7+2$$

Do you notice anything fascinating about this rearrangement? At the end of the rearrangement, we can see the addition of the digits three, five, seven, and two! This is essentially the digit sum of our original number. When we separate the digit sum we get this:

$$(3\times 999)+(5\times 99)+(7\times 9)$$
$$2997+495+63$$
$$=3555$$

Does this number look familiar? It is the exact same number we got when we subtracted the digit sum of 3572(which equals 17) from 3572! This is, in fact, divisible by nine. Which when divided elicits a number that can be multiplied by nine to get back to the original number. Let us see what happens when we plug this number(3555) into one of our terms given that x=395 as we had before:

$$(k-1)(395)=3555$$
$$k-1=9$$
$$k=10$$

We can see that “k” is equal to 10 which, therefore, means that we are in base 10. This is true because that was the original constraint given to “k” to define “x.” We could play with this all day, and possibly for eternity because this works for any natural number given that we are in base 10(for now). This brings up the point and observation that nine is very important in such simple, yet elegant patterns seen within the very nature of natural numbers. Any natural number minus its digit sum will always be divisible by nine. This means that the number nine will always be embedded within our number system. While this is a simple pattern it is nevertheless fascinating to say the least.

To conclude, I would like to propose a question. Was Nikola Tesla right about nine being one of three keys to the universe? Maybe only time will tell us where else we can find these types of patterns hidden within other areas of society, science, and art. Where else in the world does nine reveal itself in breathtaking patterns seen in the systems and structures we have created? For if we are in base ten, all natural numbers can be expressed as the sum of its digits plus some number that is always divisible by nine. It is everywhere in our number system and I suppose we could end by establishing a goal to figure out this conspiracy of mathematics, and what the other keys tell us about the world. To know that even our natural numbers, which have been around for thousands of years, can be expressed in the form of an elegant mathematical expression is such a beautiful concept to unfold.

Citations:

“Digit Sum.” *Wikipedia*, Wikimedia Foundation, 1 Mar. 2022, https://en.wikipedia.org/wiki/Digit_sum.

“Casting out Nines.” *Wikipedia*, Wikimedia Foundation, 4 Mar. 2022, https://en.wikipedia.org/wiki/Casting_out_nines#Limitation_to_casting_out_nines.