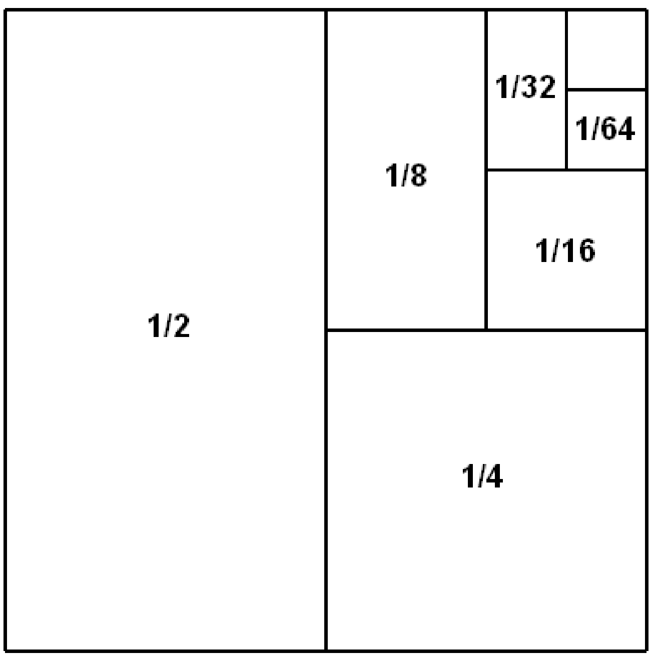
Harmony

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

As each successive term is added, the value becomes closer and closer to a particular number, in this case, 1. Since this series *converges* to a fixed value, it is a *convergent* series.



[*A visual 'proof' of this example can be illustrated using a square. The area tends towards a full square, hence the series tends to 1.*]

Meanwhile, a series that does not converge is known as a divergent series, which does not tend to any specific number (usually ∞ or -∞). Take this series, for example:

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + 16 + \cdots$$

The sum approaches an increasingly larger number, not a fixed value.

Since convergent series tend to a fixed number, can we say that:

$$\text{If } \sum a_n \text{ converges then } \lim_{n\to\infty} a_n = 0.$$?

Let's prove this statement.

Suppose we have a convergent series, and write its partial sums from 1 to n, and 1 to $n - 1$.

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_{n-1} + a_n$$

and

$$s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_{n-1}$$

a_n can then be defined by

$$a_n = s_n - s_{n-1}$$

Since we know that $\Sigma\, a_n$ is convergent, its sequence s_n must also converge. In other words: $\lim_{n\to\infty} s_n = k$, for some fixed k. We also know that if $n \to \infty$, then $n - 1 \to \infty$

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = k - k = 0$$

Hence we have proved the above statement. It is important to note that the reverse is not *necessarily* true - if $\lim_{n\to\infty} a_n = 0$, it does not necessarily follow that the series is convergent.

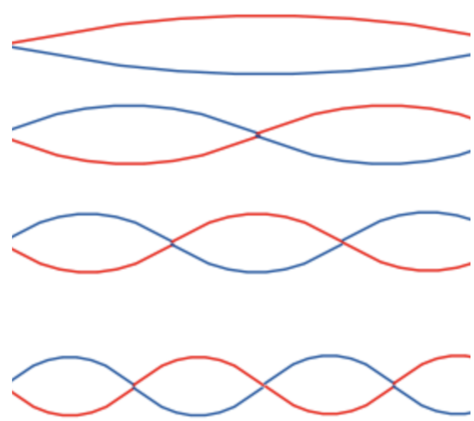
Now, from the above result, we can therefore say that:

$$\text{If } \lim_{n\to\infty} a_n \neq 0 \text{ then } \sum a_n \text{ will diverge.}$$

One particularly interesting divergent series is called the *harmonic series*:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

It is formed from the sum of every positive unit fraction. Notably, due to its divergence, it was used as part of Euler's proof of infinite primes, however I want to explore its relation to waves, and by extension, music. All objects produce a natural frequency, which can be heard when struck. When an object experiences specific frequencies, known as harmonic frequencies, a regular, repeating, wave pattern occurs - this is resonance. There are a number of natural frequencies that can produce resonance, and within this set, each frequency is related by a whole number ratio. Since instruments vibrate in this same way, it is partly why they simply sound so pleasant. Consider the vibrations produced on a string of fixed length.



The longest possible wave that can fit is known as the *first harmonic*. The next is the *second harmonic*, and so on. Regardless of the length of the string, the same pattern emerges - the second harmonic is half of the first, the third harmonic is one third of the first, continuing on in this harmonic series. The wavelength, λ, can be written in terms of the length of the string, L, for each harmonic. $\lambda = 2L$, $\lambda = L$, $\lambda = \frac{2}{3}L$, $\lambda = \frac{2}{4}L$, respectively. By taking a factor of $\frac{1}{2}$, we can clearly see the familiar harmonic series.

Amazingly, it was in fact Pythagoras who observed these results in 500 BC, by producing sounds with strings of varying length. He defined different pitches of notes by their division of the full string. Dividing the string in two would produce a tone exactly one octave above the original. Dividing further and further produced a harmonic system. Pythagoras experimented with the frequencies more, especially noting combinations that sounded pleasing. 2:1, 4:3 and 3:2 are notable for being the octave, perfect fourth, and perfect fifth, respectively. Hearing such combinations is commonplace now, but it is a testament to the importance of harmonics that we are able to decipher the very reason we enjoy them.

A Problem

Imagine you wish to cross a desert 400km from your base camp. Your jeep, however, can only carry 300km worth of fuel. Base camp has unlimited access to fuel, and you can create points along the journey where you unload any amount of fuel to be used later, then have the option to return to base to refuel again.

Create a possible journey to cross the desert with these conditions.

With some trial and error, you may have found a way to tackle this. For example, you might have created some unloading points along the way, returning each time, then completing the journey with ease. Now, think about your solution again. Is it *optimal*?

Try to find the perfect solution to this problem.

You will find that this can in fact be done with just one unloading point!
Solution:

- Travel 100km and unload 100km worth
- Travel 100km back to base and refuel
- Travel to the unloading point, refuel, then reach the destination

The *Desert Problem,* or *Jeep Problem,* has been adapted several times since it was conceived by NJ Fine in 1947. Variations include minimising the fuel needed for a given distance (as above), or maximising the distance you can travel before finishing at base (effectively making a journey there and back). Of course, the problem can be generalised.

Let n be the number of trips and k be the current trip. Suppose the jeep can carry up to 1 unit of fuel and travels 1 unit of distance per 1 unit of fuel.

- For the first trip, the jeep travels $\frac{1}{2n}$, leaving $\frac{n-1}{n}$ units of fuel.
- This leaves $\frac{1}{2n}$ units of fuel to be used to return back.
- Every $n - 1$ trips the jeep refuels fully by taking $\frac{1}{2n}$ units from the first unloading point, and takes this same amount when returning.
- For the second trip, the jeep travels $\frac{1}{2n-2}$, leaving $\frac{n-2}{n-1}$ units of fuel.
- This leaves $\frac{1}{2n-2}$ units of fuel to return to the first unloading point, which gives enough to return to the start.

- Every $n - 2$ trips the jeep refuels fully by taking $\frac{1}{2n-2}$ of fuel from the second unloading point, and takes this same amount when returning.
- For the k^{th} trip, the jeep needs to create an unloading point such that it is $\frac{1}{2n-2k+1}$ units of distance from the $(k - 1)^{th}$ unloading point.
- Leaving $\frac{2n-2k-1}{2n-2k+1}$ units of fuel
- Therefore, every $n - k - 1$ trips, it refuels fully by taking $\frac{1}{2n-2k+1}$ from the k^{th} unloading point, and takes this same amount when returning.

Finally, consider the final trip:
- There are, in total, $n - 1$ unloading points.
- Starting from the deposits closest to the destination, the fuel left is:

$$\frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2n-1}$$

Therefore, if we include the starting unit of fuel for the final trip, the total distance is:

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$$

This is a form of the harmonic series, the *odd* harmonic series.

If we had considered the objective to be returning back to base as the final trip, for the k^{th} trip, the unloading point would be $\frac{1}{2n-2k+2}$ units away from the previous point, instead. Following the same process as above, our maximum distance into the desert would be:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}$$

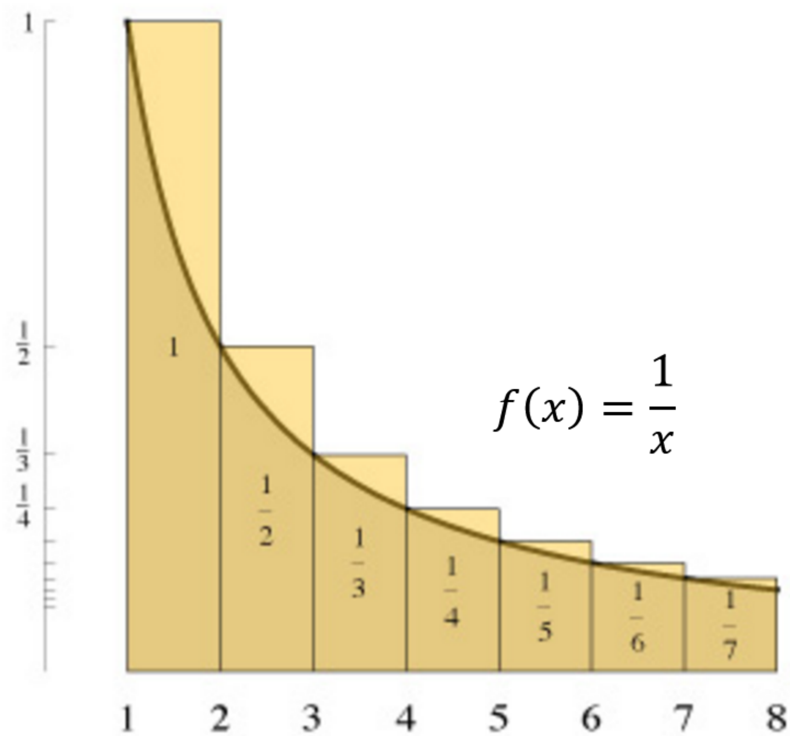
The total distance travelled, however, is twice this, giving:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

The harmonic series.

But what does this mean for the problem? Well, since we know that both the harmonic and odd harmonic series are divergent, tending towards ∞, any distance can be travelled (as long as you are willing to take the necessary number of trips!).

Using the connection to the harmonic series, we can even estimate the number of trips needed in order to travel a certain distance.



This is a geometric interpretation of the harmonic series, with the line $f(x) = \frac{1}{x}$ superimposed. As you can see, the area under the line is approximately equal to the sum of the harmonic sequence. In other words:

$$\int \frac{1}{x} \approx \sum_{k=1}^{\infty} \frac{1}{k}$$

Hence:

$$ln(|k|) \approx \sum_{k=1}^{n} \frac{1}{k}$$

Suppose we wanted to travel a distance of 3 units (same rules for the jeep apply).
Plugging in 3, we have:

$$ln(x) = 3$$
$$x = e^{3} \approx 20\ trips\ needed$$

This value increases exponentially; take a distance of 8, say.

$$ln(x) = 8$$
$$x = e^{8} \approx 2980\ trips\ needed$$

If the objective is to return back at the end, the farthest distance into the desert is only half the total distance travelled, and so we have:

$$\frac{1}{2} ln(|k|) \approx \sum_{k=1}^{n} \frac{1}{k}$$

Using the same examples, for a distance of 3 units, we have:

$$\frac{1}{2} ln(x) = 3$$
$$ln(x) = 6$$
$$x = e^{6} \approx 403\ trips\ needed$$

And

$$\frac{1}{2} ln(x) = 8$$
$$ln(x) = 16$$
$$x = e^{16} \approx 8886110\ trips\ needed!$$

The harmonic series is inherently fascinating, yet its properties and appearances discussed here are just a portion of its use. Experiencing harmony may seem elementary to us now, but rooted in its simplicity is truly beautiful mathematics.

Sources:

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