Gödel’s Incompleteness Theorem, Axioms, and Mathematical Epistemology

What is an axiom, and why do they matter?

Mathematics, like the world we inhabit, suffers from the epistemological ailment that the rest of the observable and unobservable world does - how can we know anything? In mathematics, this manifests in questions about proof, and how can we *know* statements to be true. Unlike the natural sciences, we do not deduce knowledge of mathematical reality through theory or observation, and instead use the tools of reason and logic. However, to be able to prove a statement, only previously proved statements can be used. Like the famous “which came first, the chicken or the egg” problem, there must logically be a “first” true statement, which on the basis of, all other statements can be proven. In the field of ontology, by doubting everything he thought he knew, Descartes claimed this statement to be “I think therefore I am”. Mathematics does not have the luxury of a beginning statement discernable solely from reason, so the idea of an “original” truth breaks apart. As any statement requires a previous one to prove it, there can be no original.

Mathematician’s solution to this problem may seem like a crude one - it was just to proclaim that some statements are unprovable but true, and to construct the rest of mathematics off of them. These statements are called “axioms”, and from them, in theory, you can prove any statement in mathematics on top of them, like the foundations of a house. Any proof constructed with these would be described as axiomatic proofs, as according to the current axioms they must be and always will be true. The artificial synthesis of these axioms may lead some to have doubts about the validity of mathematics - how can we know that our version of mathematics is ‘correct’ if we have just chosen the axioms to prove it all?

Mathematics is a tool we use to understand the world, and we do this by abstracting our reality. The axioms mathematicians have created are from breaking down these abstractions into fundamental rules that will always apply in the real world. The only ‘correct’ Mathematics is what we decide it to be, and for most this is based on what we observe. This is why axioms are important - they are the feature that makes mathematics a practical quantitative tool. This is where mathematical and philosophical axioms diverge.

These axioms fall into two categories: the logical and non logical. This is the difference between the tautological axioms, and the ones that are more specific to a field. Logical axioms include the axioms of equality, whereas non logical axioms are typically about operations.

$x = x$

If $x = y$, $y = x$

If $x = y$ and $y = z$, then $x = z$

These axioms are tautological, meaning they are true by definition. The way we define numbers is fitted to these seemingly obvious rules; an object that did not would probably not be a ‘number’. One cannot really envisage mathematics without these being true.

Some examples of non logical axioms relate to addition and multiplication.

$x = y$ and $z = w$, then $x + y = z + w$

If $x = y$ and $z = w$ then $xz = yw$

These, while they may still seem obvious, are not *tautologically* true. One could have a mathematical system where these were not true, even if it would not accurately model the real world.

Gödel’s Incompleteness Theorem

At the turn of the 20th century, Mathematician David Hilbert proposed a set of 23 problems that mathematicians should attempt to solve within the following century. The ‘theme’ of the problems laid out were based on Hilbert's wish to unite and formalise mathematics; principal among these being the wish to “arithmatise analysis”. This entailed for mathematics to become independent of intuition, and for all concepts within mathematics to be explained using fundamental axioms. Of most retrospective significance, was his second problem. It challenged mathematicians to prove that all mathematics was free of contradiction, and that it was consistent within its own rules.

Much to the disappointment of Hilbert, this conjecture was proven false by an Austrian Logician Kurt Gödel. He put forth two theorems of incompleteness, or ways in which mathematics is and never will be fully consistent or understood. The essence of his first proof is that regardless of the axioms of any mathematical system, there will always be statements that are unprovable using those axioms. The second is that it is impossible to prove whether there are logical inconsistencies within any mathematical system with a given set of axioms using said axioms, however may be possible using a different expanded set of axioms.

This discovery completely shattered the worldview of all mathematicians, but especially the disciples of Hilbert. Their vision of a unified and comprehensive standard of mathematics was no longer on the horizon - and problems that were fundamental to mathematics, that some had dedicated their life to proving, now had the possibility of being impossible to prove. What made the job of mathematicians even more difficult, is that it was also shown that you cannot prove whether a statement is unprovable or not - as that would be a form of proof.

From a philosophical standpoint, there are also parallels to be drawn from the realm of the unexperimentable. If a statement is unfalsifiable, then it is unable to be shown to be false (or true). Classic examples often relate to the metaphysical or supernatural - questions such as the existence of a god or whether or not we live in a simulation. One can look at these the same way as in mathematics - we do not have the correct “scientific” axioms to reason these to be true or false. Not just that we are unsure, but we physically do not have the tools to be able to solve that category of problem. Unlike the idealised landscape of mathematics, we cannot add or change the nature of the universe we inhabit and the rules it follows, so these statements can never be proven. In this respect, Gödel’s theorems almost become logical - how can we expect a system based off of the real world to not have its flaws and limitations?

How did Gödel prove this?

One question that often is raised among those hearing about Gödel’s revolutionary work is how he managed to prove it. Unlike many other mathematical proofs, it is not an equation or a relationship that can be described and solved, and it is also self referential - it is a proof about what is possible about mathematics, not just an idea within it. To anyone with knowledge of mathematical proofs, how one would even tackle a problem that is so distanced from traditional methods. The way he went around proving this was through an ingenious method of defining each possible statement in mathematics, regardless of whether it was true or false. Each operation and object within any possible statement was given a prime number value, and then a unique value was generated from the product of each element of the statement.

In natural language, much to the fascination of toddlers, statements such as “This sentence is false” exist. It is contradictory by nature, as if it were to be true, it must be false, and vice versa. It cannot be rationalised to have any logical sense. These have very little impact on our ability to communicate, and do not imply some fatal flaw in our primary method of communication, so these quirks are mostly just ignored as an oddity. Language is not a system that is fully consistent or logically sound all the time. What Gödel managed to do, is to construct a statement using his system in a similar vein to those language statements. ‘This statement cannot be proven using arithmetic’, if proved, would contradict the system in which it inhabits. Since Mathematics, especially proofs, must be logically consistent, this must mean that there are some statements which can never be proven using arithmetic.

The Importance of Philosophy for Mathematics

It is very tempting, for us as mathematicians, or even those with just a passing interest in maths, to write off philosophy as unimportant to our field of study. To some, they may seem on opposite ends of a spectrum. Mathematics is the most quantifiable field of inquiry; the tools mathematics provides are adaptable and useful for solving any real world problem involving quantities and relationships of systems. Philosophy is the near polar opposite - a subject devised on the study of ideas, removed from reality, and lacking in any quantifiable aspects that mathematics lends its hand to. However, I would argue these are just two sides to the same coin.

Both mathematics and philosophy are abstractions of concepts in the real world. Philosophy is about the abstraction of ideas, and mathematics about the abstraction of relationships. While the mediums they deal with abstract concepts are different, the way in which they handle them is highly similar. The rules of logic that were developed originally by Ancient Greek philosophers (who were also mathematicians) can be applied to both disciplines, and arguably the fundamental ways in which concepts are handled in both are similar. Understanding philosophy, how to reason, and in particular epistemology or the study of how we can know is of great usefulness to us mathematicians. The subjects are not only similar, but interlinked; they only solve problems of different sorts with the same tools. Ideas from philosophy and linguistics about proof are integral to our ability to reason as mathematicians, as I hope I have shown, as it was their tools that got us to where we are. The exchange of concepts between the fields is hardly over, and likely will be used again and again.

Oliver Summerfield

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