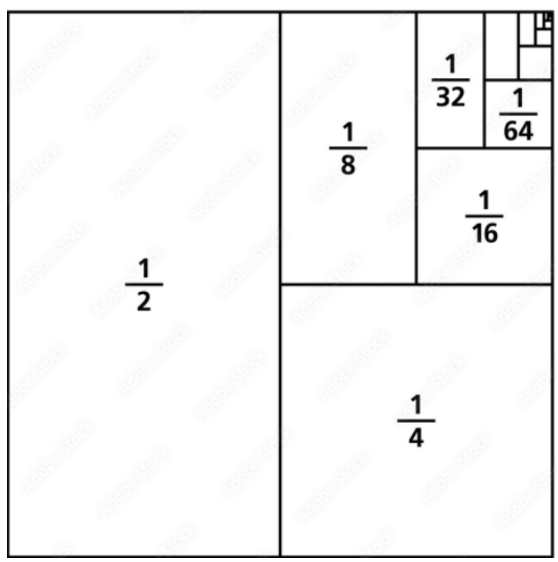
I've always been fascinated by the notion of infinity and especially by infinite sums - the idea that you can add infinitely many things together and yet end up with a finite often simple number. I was first shown this by my teacher through Zeno's paradox of motion. Zeno says that in order to reach some place P, at some point you must have travelled halfway to P so half the journey remains. Then later, you must have travelled half of that journey so in total a quarter of the journey remains, then repeating, an eighth will remain and so on. As we are able to reach P and during that journey we have travelled half of the way, then another quarter, then another eighth, then another sixteenth... we can say

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\ldots=1$$

Zeno was stumped by the concept that infinitely many finite distances can combine to form another finite distance and hence called this a paradox, but really he was describing convergence.

That expression also has a really nice geometric interpretation shown below of a square of side length 1 built up of all these areas in the sum.



And in general for any decreasing infinite series with a common ratio, there is a nice formula to evaluate it. We start with a general series with first term a and common ratio r with $\mid r \mid < 1$:

$$S=a+ar+ar^2+ar^3\cdots$$

$$S = a + r(a + ar + ar^2 + ar^3 \dots)$$

$$\Rightarrow S = a + rS$$

By rearranging we get the result $S = \frac{a}{1-r}$. These types of infinite series are called geometric progressions.

However, this formula then invites the question of what if the ratio between terms isn't constant? How can we evaluate those sums and do they also always converge? To answer this, let's look at another type of infinite series which has some more curious features - the harmonic series:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \dots$$

When you first play around with this sum, you might think that this sum quickly converges after adding up a few terms but surprisingly, it will increase forever. This was first proved by Nicole Oresme way back around 1350 but here I'll present a more recent and slick proof by Honsberger in 1976. He uses the classic method of proof by contradiction.

First suppose that the harmonic series does converge to some real number, say X.

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \dots = X$$

We can now pair up the fractions and create an inequality as the first fraction in the pair is always bigger than the second.

$$X = \left(\frac{1}{1} + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10}\right) \dots >$$

$$\left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{10} + \frac{1}{10}\right) \dots$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \dots = X$$

$$\Rightarrow X > X$$

This is clearly a contradiction as any finite real number cannot be greater than itself and so it must be that the harmonic series diverges.

Even though every term is getting smaller and every term is ≤ 1, as the number of terms being added on tends to infinity the sum still diverges. Isn't that remarkable? When I first came across this it still seemed very counter-intuitive because no matter how many terms I decided to manually add up, whether it was the first 10, 20, 100 or even 1000 terms (via python), the sum never got very large and for those values it wouldn't even exceed 10. It turns out that it diverges incredibly slowly, taking more than 10^{400} terms for the sum to even reach 1000 which on the scale of the infinite number line is minuscule. That's far more terms than there are atoms in the observable universe!

Naturally, the next step is to bump up the powers and try the sum of the reciprocals of the squares. This however, is much harder to evaluate and is known as the Basel problem. It was unsolved for 90 years until Euler came along and proved that it converges to $\frac{\pi^2}{6}$ in 1735 which brought him into fame. This result is even more baffling because it has π in it and not only is π present when there seems to be no relation to circles but it's squared which is rarely seen in any formulas to do with circles or curved objects. This is probably my favourite infinite sum because it's just so unexpected and mysterious and so I'm going to pick apart Euler's solution.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} \cdots = \frac{\pi^2}{6}$$

Euler's ingenious proof bases on the idea of representing the sinc(x) function as an infinite degree polynomial in two different ways and comparing coefficients such that this sum just pops out magically. He starts with the Maclaurin series for sinc(x)=$\frac{\sin x}{x}$ which was well known at the time.

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} \cdots$$

That was the first way of representing the function. Any continuous function in an interval [a,b] can be represented as closely as desired with the sum of polynomials (this result was actually unproved at the time) and so essentially he now wants to represent sinc(x) as another infinite degree polynomial.

If the polynomial is going to be equivalent to sinc(x), it must have the same roots. The roots of sinc(x) are the same as that of sin(x) which are $\pi, -\pi, 2\pi, -2\pi, 3\pi \cdots$. By the factor theorem, if c is a root of a polynomial, then (c-x) is a factor so now we can try and construct this polynomial. We can consider -

$$(\pi - x)(\pi + x)(2\pi - x)(2\pi + x)(3\pi - x)(3\pi + x) \cdots$$

This is close but in the Maclaurin series the constant term is 1, but here we would have it being infinite so we need to reduce the magnitude of our factors to get the constants in each bracket to multiply to 1. We can do this by making all our factors have a constant term of 1 in it so they all look like (1...)(1...)(1...)(1...)...

Therefore we must divide the first pair of factors each by π, the next pair by 2π, the next pair by 3π and so on. Euler puts this all together and expresses sinc(x) as -

$$\frac{\sin x}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots$$

Every pair of factors can be combined using the difference of squares formula $(x+y)(x-y) = x^2 - y^2$ to give

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\left(1 - \frac{x^2}{25\pi^2}\right)\cdots$$

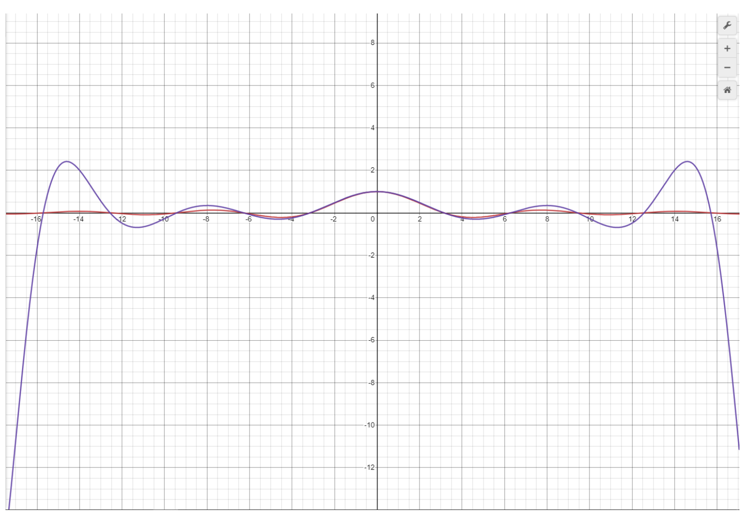
The Maclaurin series and this infinite polynomial must be equivalent for all x as they are both sinc(x) therefore we can compare and equate coefficients and here Euler chooses to do so for x^2.

$$-\frac{1}{3!} = -\frac{1}{\pi^2}\left(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}\cdots\right)$$

Rearranging gives the miraculous result -

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}\cdots = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{3!} = \frac{\pi^2}{6}$$

Sidenote: Here is a graph comparing sinc(x) (RED) with just the first 6 terms of our infinite polynomial (PURPLE) - the approximation is especially good close to x=0 and will continue to improve as more terms are used.



However, the elegance of this method comes from the fact that it is extendable and we could also choose to compare different coefficients and they would also be equal such as for x^4.

$$\frac{1}{5!} = \frac{1}{\pi^4}\left(\frac{1}{1}*\frac{1}{4}+\frac{1}{1}*\frac{1}{9}+\frac{1}{1}*\frac{1}{16}\cdots+\frac{1}{4}*\frac{1}{9}+\frac{1}{4}*\frac{1}{16}+\frac{1}{4}*\frac{1}{25}\cdots+\cdots+\cdots\right)$$

The contents of the brackets on the RHS is the sum of all pairs $1/(ab)^2$ for any natural numbers a and b excluding all cases where $a = b$ as no term appears twice.Therefore we can write it as -

$$\frac{1}{5!} = \frac{1}{\pi^4}\left(\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{(ab)^2} - \sum_{a=1}^{\infty}\frac{1}{(a*a)^2}\right)*\frac{1}{2}$$

We have divided by two on the RHS because the sum of the terms with a=1,b=2 for example is the same as the sum of the terms with a=2, b=1 by symmetry so we have over counted by a factor of 2. The sum of a sum is the same thing as multiplying the individual sums so we can rewrite the expression on the right to be -

$$\frac{2}{5!} = \frac{1}{\pi^4}\left(\sum_{a=1}^{\infty}\frac{1}{a^2}\sum_{b=1}^{\infty}\frac{1}{b^2} - \sum_{a=1}^{\infty}\frac{1}{a^4}\right)$$

From before, we have calculated $\sum_{a=1}^{\infty}\frac{1}{a^2}$ to be $\frac{\pi^2}{6}$ and so we can now evaluate $\sum_{a=1}^{\infty}\frac{1}{a^4}$ just by rearranging:

$$\sum_{a=1}^{\infty}\frac{1}{a^4} = \left(\frac{\pi^2}{6}\right)^2 - \frac{2\pi^4}{5!} = \frac{\pi^4}{90}$$

With little extra work we have already calculated another difficult infinite sum and this method works for any even power so Euler's solution can be used to evaluate all sums of the form $\sum_{k=1}^{\infty}\frac{1}{k^{2n}}$!

Not only did Euler manage to solve that infamous problem, but he also provided a very useful factorisation of all series of this type. Consider the expansion of this product:

$$\left(\frac{1}{1}+\frac{1}{2^n}+\frac{1}{2^{2n}}+\frac{1}{2^{3n}}\cdots\right)\left(\frac{1}{1}+\frac{1}{3^n}+\frac{1}{3^{2n}}+\frac{1}{3^{3n}}\cdots\right)\cdots\left(\frac{1}{1}+\frac{1}{(p_k)^n}+\frac{1}{(p_k)^{2n}}+\frac{1}{(p_k)^{3n}}\cdots\right)\cdots$$

Here the number of brackets are the number of distinct primes. Each bracket contains a geometric progression, like we saw earlier, with a common ratio of $\frac{1}{(p_k)^n}$ for some prime p_k and first term 1. Therefore each bracket can actually be represented more succinctly by the expression $\frac{1}{1-\frac{1}{{p_k}^n}}$

We know also that every natural number x has a unique prime factorisation, and is made up of the product of distinct primes each to some power - x=

$p_1^{a_1}p_2^{a_2}p_3^{a_3}\cdots p_m^{a_m}$. So by extension, x^n is just the product of all those primes each to n times of their original powers: $x^n = p_1^{na_1}p_2^{na_2}p_3^{na_3}\cdots p_m^{na_m}$

Therefore when we expand out the brackets, because all the primes and every possible power of that prime of the form p_k^{-na} is present where a is any non-negative integer, it actually expands out to form x^{-n} for all natural numbers x. It would be a good exercise for the reader to check properly that you really do get the below expression!

Therefore, this product, known as the Euler product expands to be:

$$\prod\left(\frac{1}{1-\frac{1}{{p_k}^n}}\right) = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \frac{1}{9^n}\cdots$$

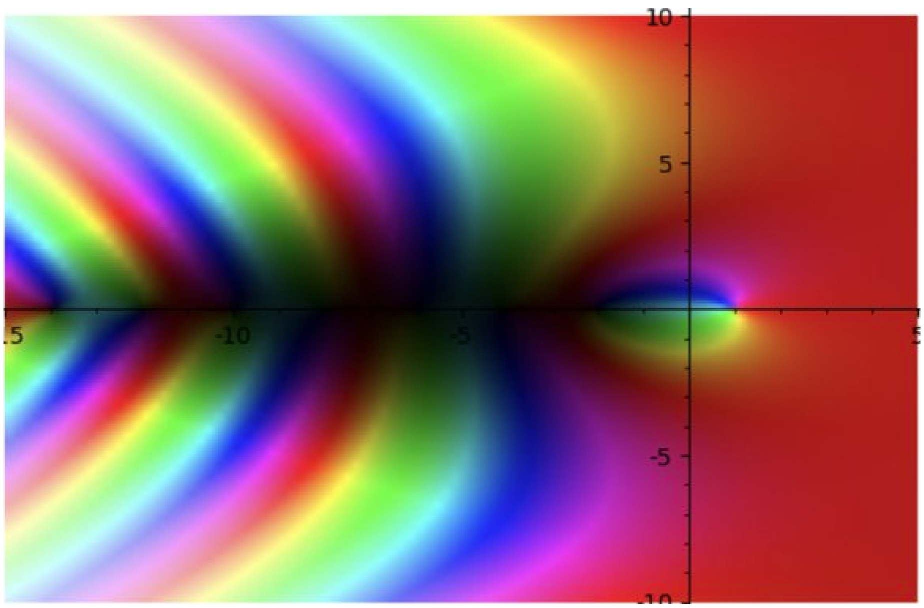
To explain this a little more clearly I'll use a couple of examples of how each term has been formed: The $\frac{1}{6^n}$ has been formed by multiplying $\frac{1}{2^n}$ from the first bracket with $\frac{1}{3^n}$ from the second bracket and then 1s from every other bracket. Similarly, the $\frac{1}{8^n}$ has been formed by multiplying $\frac{1}{2^{3n}}$ from the first bracket with 1s from every other bracket. And so on.

Why is this at all useful I hear you ask?

While these sums and products are fun and interesting, the Euler product and these infinite series also provide many helpful and unexpected results. One such result is a neat proof that there are infinitely many primes.

This product where n=2 as we proved earlier is equal to $\frac{\pi^2}{6}$ and as π is irrational, so is $\frac{\pi^2}{6}$. Each bracket in the product is $\frac{1}{1-\frac{1}{{p_k}^2}}$, which is equivalent to $\frac{p_k^2}{p_k^2-1}$ by rearranging and hence is rational. The product of finitely many rational terms together is itself rational but here the result of our product is irrational so it must be that there are infinitely many terms. Since each term in the Euler product contains a distinct prime there must be infinitely many primes. I think this is a really nice proof because despite the prerequisites, every step feels quite intuitive in comparison to Euclid's classic proof.

Well again, that was a nice result but it was already known, what about current maths? It turns out that these sums are very relevant for ongoing mathematics as these sums we've calculated are special cases of the Riemann Zeta function, a function regarded by many as the most important in all of mathematics. The Riemann Zeta function, created by Bernhard Riemann in the 19th century, is essentially $\zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n} = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n}\cdots$ defined over the complex plane with the real part greater than 1. When the real part is not greater than one, the function is defined through something called analytic continuation. Surprisingly, though Euler evaluated all even positive integer values of the Zeta function 300 hundred years ago, all odd values > 1 are still not known precisely today! This function has been heavily studied by mathematicians for over a hundred years in the pursuit of proving the Riemann Hypothesis which essentially describes when the function will 'non-trivially' go to 0. This is very important because the Riemann Hypothesis allows for very precise predictions about the distribution of primes. There is even a million pound prize for anyone who can prove or disprove it! The maths involved is far too complex for me to understand so I just wanted to touch upon it here but it is also really quite beautiful: this is the Riemann Zeta Function in the complex plane with domain colouring - The colour represents the argument of the output increasing arguments from red to blue and the brightness being the magnitude, with brighter values being 'bigger'.



Infinite sums continue to challenge mathematicians, from those in secondary school to world class number theorists, and yet, they never fail to amaze, confuse and still be profoundly satisfying.

References:

All images are screen captions and used for educational purposes.

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