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Patterns in Primes

Introduction

Prime numbers are seemingly random in nature; they certainly do not follow the rules of a standard arithmetic, geometric or even polynomial sequence as they lack any sort of common ratio or difference. The only condition for a number to be prime is that it must not have any factors other than itself and 1 – which makes it very difficult to link primes to both other primes, as well as other numbers in general. This seemingly *random* property of primes makes both for a very interesting branch of mathematical research, which is a sub-section of the branch known as *Number Theory*, as well as practical applications, such as the use of prime numbers in cryptography and general security - as well as the use of prime numbers in art as an expression of natural phenomena.

This dissertation aims to give a brief introduction to the vast amounts of research that has gone into these special numbers, as well as the benefit this has to both mathematics and other walks of life.

Methods in Number Theory

There are several methods used in Number Theory to generate prime numbers, or prove a property about prime numbers, and mathematicians have attempted to link these different methods to find overall patterns in the prime numbers.

One of these methods is Euclid's proof – the proof that there are infinite prime numbers:

- Euclid's proof works as a proof by contradiction, as well as a direct proof by cases: First assume there is a finite list of prime numbers $p_1, p_2, p_3 \dots, p_n$.
- Let P be the product of these primes (i.e. $P = p_1 p_2 p_3 \dots p_n$)
- Now let N be $P + 1$ and now consider the primality of N:
    - If N **is** prime – a new prime number has been generated that was not in the original list of supposedly every prime number in this finite system.
    - If N is **not** prime – N must have a new prime factor. If this prime factor was in the original list, it would divide both N and P and as a result it would have to divide $N - P$ which is 1. There is no number that divides 1 and so this prime factor must not be in the list of primes.
    - Either way, a new prime has been discovered outside of the finite list of primes mentioned earlier.
- There is now a method to generate primes outside of the supposedly finite list of primes and thus by contradiction we can conclude that the set of all prime numbers must be infinite.

Euclid's proof is significant because it forms the basis of the study of prime numbers, as the idea that the list of prime numbers never terminates directly links prime numbers to our understanding of the set of natural numbers as a whole.

We can also look at a section of prime numbers known as *Mersenne primes* to identify other links to the natural numbers. A Mersenne prime is a prime number that is defined to be 1 less than a power of two, thus we can conclude that a Mersenne prime can be written in the form:

$$2^n - 1, \ \ n \in \mathbb{N}$$

Mersenne primes are significant for numerous reasons:

- The fact that they can be written in the form $2^n - 1$ is very convenient, because the binary representation of such a number is simply n 1s in a row, making them easy to compute as computers operate on a binary system – this is illustrated by the fact that many of the largest prime numbers found as of 2022 are Mersenne primes. Figure 1 shows a graph of the largest known prime number over the years.

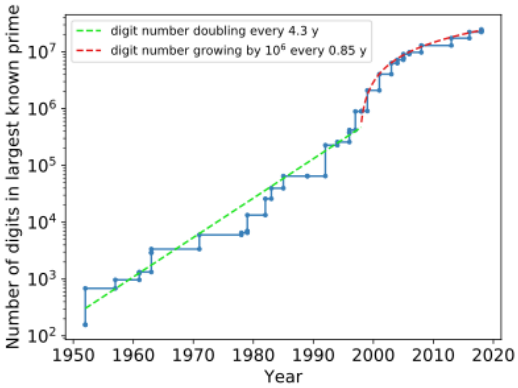

Figure 1 - A graph of the largest known prime number over the years.

- For a given Mersenne prime, $2^p - 1$, it is known that p is also prime.
    - To prove this, we use what is known as the contrapositive – in this case "If p is not prime, then $2^p - 1$ is not prime."
    - Let $p = ab,\ a, b \in \mathbb{N},\ a,\ b\ \neq 1$
    - $2^p - 1 = 2^{ab} - 1$
    - We can factorise this expression as follows:
    $$2^{ab} - 1 = (2^a - 1)[(2^a)^{b-1} + (2^a)^{b-2} + (2^a)^{b-3} + \cdots + 1]$$
    - $2^p - 1 = 2^{ab} - 1 = (2^a - 1)[(2^a)^{b-1} + (2^a)^{b-2} + (2^a)^{b-3} + \cdots + 1]$
    - $[(2^a)^{b-1} + (2^a)^{b-2} + (2^a)^{b-3} + \cdots + 1] \neq 1$
    - $2^p - 1$ is not prime as it is a composite number.
    - By contrapositive, if $2^p - 1$ is prime, then p is prime.
- Mersenne primes are connected to *perfect numbers* – these are numbers that are defined to be equal to the sum of their *proper* factors (i.e. all factors apart from the number itself). This once again links a group of prime numbers to the natural numbers. The direct link between Mersenne primes and perfect numbers is as follows:
$$(2^p - 1)(2^{p-1}) = N \text{ where } N \text{ is a perfect number.}$$
    - To prove this result we expand write out the sum of the factors of $(2^p - 1)(2^{p-1})$: $(1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{p-2} + 2^{p-1})(2^p - 1) + (2^p - 1) + 2^1(2^p - 1) + 2^2(2^p - 1) + 2^3(2^p - 1) + \cdots + 2^{p-2}(2^p - 1) + 2^{p-1}(2^p - 1)$
    - $= (1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{p-2} + 2^{p-1}) + (2^p - 1)(1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{p-2} + 2^{p-1})$
    - $= 2^p(1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{p-2} + 2^{p-1})$
    - Let $T = (1 + 2^1 + 2^2 + 2^3 + \cdots + 2^{p-2} + 2^{p-1})$
    - $2T = (2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^{p-1} + 2^p)$
    - $T = 2T - T = (2^p - 1)$
    - Substitute back in: $2^p(2^p - 1)$
    - $= 2[(2^{p-1})(2^p - 1)]$ Divide by two to get the sum of the proper factors (N)

Unsolved Problems Relating to Prime Numbers

There is lots yet to be found when it comes to prime numbers and particularly the patterns in the prime numbers. One way mathematicians decided to tackle this is by looking at the gap between prime numbers and thus the *Twin Prime* Conjecture was made by French mathematician *Alphonse de Polignac*

The Twin Prime Conjecture states that there are infinitely many primes that have a gap between them of two – namely the smallest possible gap between prime numbers (excluding the gap of 1 between 2 and 3). This can be summarised as:

$$\lim_{n\to\infty} \inf(p_{n+1} - p_n) = 2$$

What makes the Twin Prime Conjecture inherently difficult to prove is that as you reach larger and larger numbers, the average gap between two primes seems to increase as well, but it is thought to always be the case that there are twin primes as shown by Figure 2. This is thought to be the case because of a discovery made by mathematician *Yitang Zhang*, namely that:

$$\lim_{n\to\infty} \inf(p_{n+1} - p_n) < N, \quad N = 7 \times 10^7$$

While it seems that a gap of 70 million is far from the required gap to be proven of two for the Twin Prime Conjecture, this was the first major breakthrough in actually finding a consistent gap that will always occur and this N value has been further reduced to 246 by *The Polymath Project* via their optimization of Zhang's bound.

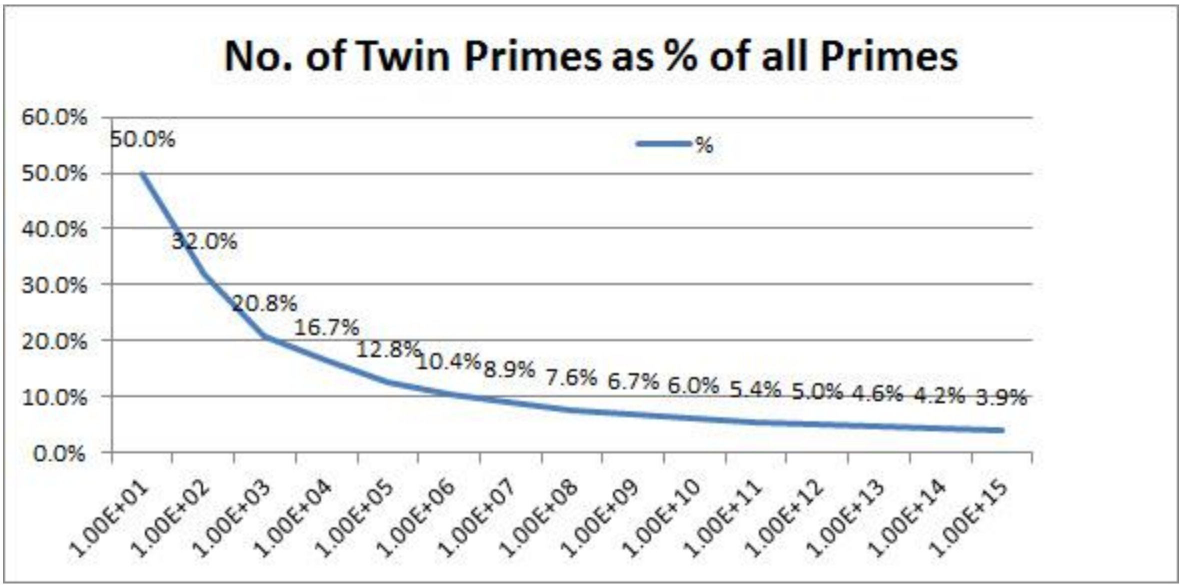

Figure 2 - Proportion of all primes that are twin primes.

Another problem that can be linked to the distribution of prime numbers, albeit in a less clear way, is the very famous *Riemann Hypothesis*. The Riemann Hypothesis is a conjecture made by *Bernhard Riemann* and is based on the *Riemann Zeta Function* $\zeta(s)$ where:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

The Riemann Zeta Function famously has zeroes at $s = -2n,\ n \in \mathbb{N}$ and these are known as *trivial* zeroes. The Riemann Hypothesis, however, states that all non-trivial zeroes lie on a *critical strip* – this strip being $s = \frac{1}{2} + ti,\ t \in \mathbb{R} \cap i$ is the imaginary unit.

The Riemann Zeta function is a complex function which is only well defined for $s > 1$. To truly unlock the full Riemann Zeta function, analytical continuation must be used (see Figure 3) and so it is hard to believe that such a conundrum is linked to the prime numbers, when prime numbers are purely discrete real numbers and the Riemann Hypothesis is inherently based on calculus and complex analysis.

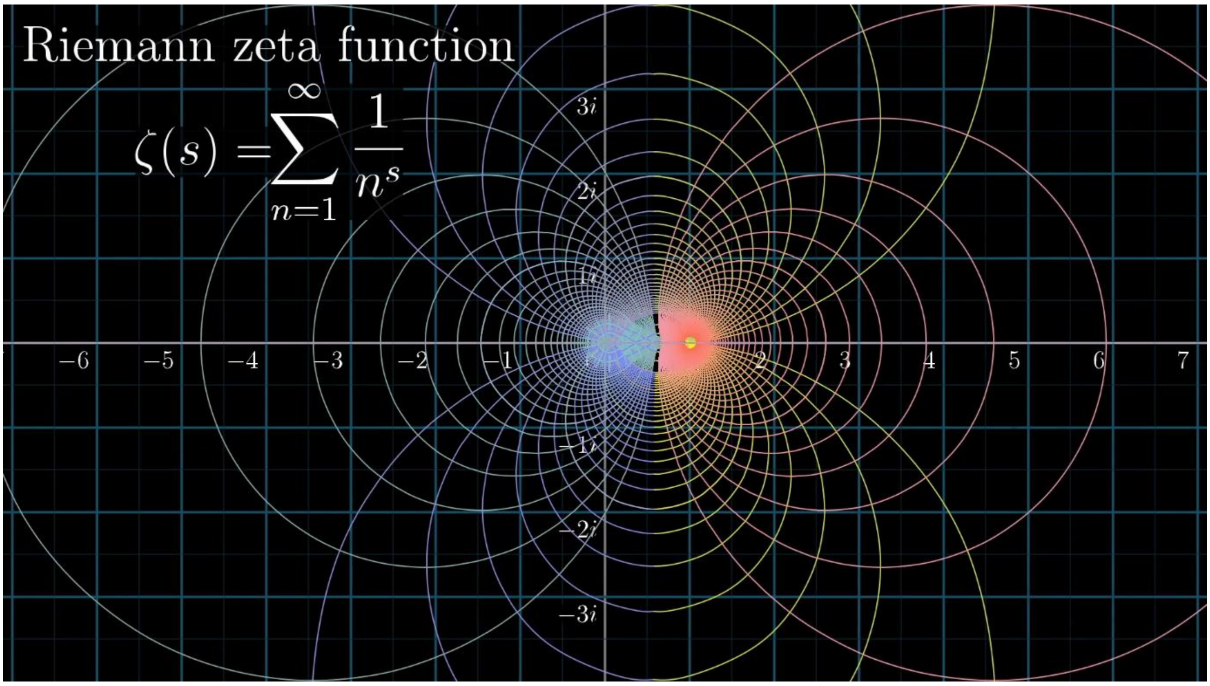

Figure 3 - The Riemann Zeta mapping showcasing analytic continuation

The relationship between the Riemann Hypothesis and the distribution of prime numbers is incredibly complicated, but it is to do with the *prime number theorem* which formalizes the previously mentioned idea of the gap between prime numbers increasing as the numbers you are looking at increase.

Applications of Prime Numbers

While the study of prime numbers does primarily come from the development of our understanding in Number Theory, there are a number of useful applications of prime numbers that can be reinforced by the rigorous study of them.

The aforementioned use of prime numbers in cryptography because of their seemingly random nature is a significant application. It stems from the *RSA Algorithm* developed by *Ron Rivest, Adi Shamir* and *Leonard Adleman* in 1977, which allows a message to be encrypted without the sender knowing the key. This arises from the fact that if you multiply two prime numbers together, those two prime numbers are going to be their only factors.

Prime numbers are also used to simulate natural phenomena in art, allowing artists to create very naturally feeling works. It is also significant that prime numbers appear in nature, such as the hibernation of *Cicadas*. These insects emerge every 13 or 17 years from hibernation to mate – most likely to avoid predators.

Conclusion

In conclusion, the study of prime numbers is very significant to developing our understanding of Number Theory and we can see this through both the rigorous proofs and results discovered by mathematicians centuries and even millennia ago, as well as through the current effort to solve incredibly complex problems such as the Riemann Hypothesis and the Twin Prime Conjecture. The study of prime numbers and their patterns is also significant due to our applications of prime numbers in cryptography, art and even areas in science such as biology.

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