REGULAR POLYHEDRA IN HIGHER DIMENSIONAL EUCLIDEAN SPACE

I want to write about regular polyhedra that aren't platonic solids - this is something that is not talked about very much. It contains some new maths that I think anyone reading this would like.

I don't know if I'm mentioning all the regular polyhedra in higher dimensional space, but these are all the ones that I know about. I have learnt a lot of this information from Jan Misali's videos about regular polyhedra on YouTube.

We will first start by talking about polygons - the building blocks of polyhedra.

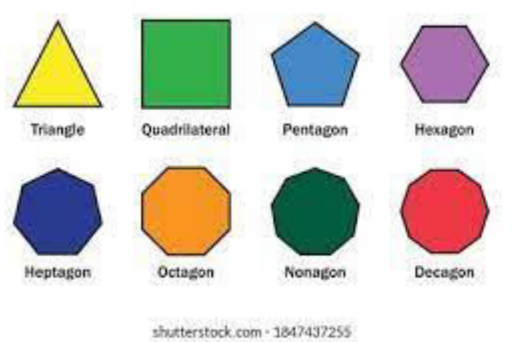
There is a category of polygons called regular polygons. The majority of information on the internet says this, "A polygon is regular when all angles are equal and all sides are equal (otherwise it is "irregular")".

Retrieved from https://www.mathsisfun.com/definitions/regular-polygon.html.

However, there are some people out there, and I am one of them, who believe that the definition of a regular polygon is a polygon that is vertex transitive and face transitive. I will now explain what that means. A polygon is edge transitive if you can move one edge into another edges position without changing how the polygon looks. Vertex transitivity is the same thing, but with vertices (aka corners).

I also want to clarify something - the inside of a polygon or polyhedron is not part of that polygon or polyhedron - remember this, it will be useful later.

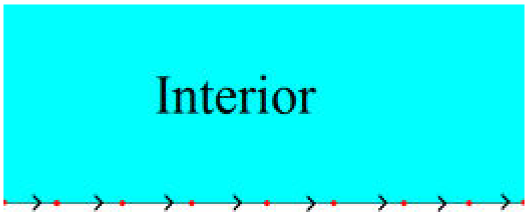
Did you know that the regular polygons that you learn in school are only a fraction of the regular polyhedra that are out there?. These are some of the regular polygons that you learn in school:



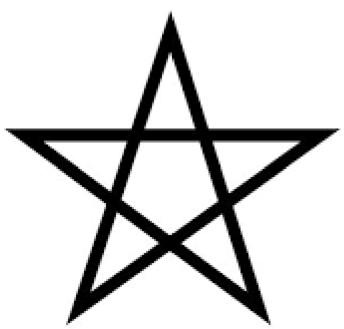
This picture consists of a three-sided shape, a four-sided shape, a five-sided shape...ten-sided shape. Sometimes you also learn about all the polygons with more than ten sides such as hendecagon, dodecagon etc.

The following is something that I believe - that a two sided shape is legal. I know that this is against popular opinion. Imagine a two-sided shape. To a so-called "sensible" mathematician, a two-sided shape makes no sense - it would just be two vertices connected by two line segments, but those two line segments would be in the exact same place and it would just look like a single line segment, so that's why they don't allow it. However I do, because a two-sided shape (called a digon) would have zero area, and not calling a two-sided shape legal because it has zero area is just a case of discriminating against zero.

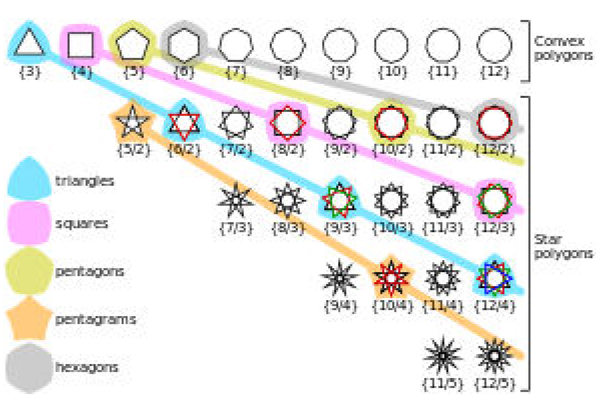
The following is is not controversial - you can have regular polygons with infinitely many sides like this apeirogon.



Remember, on all of these pictures and all of the pictures that I will show you, the interiors of the shapes are not part of the shapes themselves. Also, you can have polygons that intersect themselves like this pentagram below. Thankfully there is no definition for an inside and an outside of a pentagram, so people don't start believing that the interior of a pentagram is part of the pentagram itself.

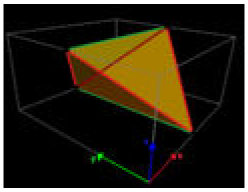


The pentagram is a stellated pentagon, because it is a pentagon where you extend the edges until they meet up again and the intersection points are not vertices. The pentagram is a regular star polygon. Below are some of the other regular star polygons as well as the common regular polygons (called convex regular polygons).

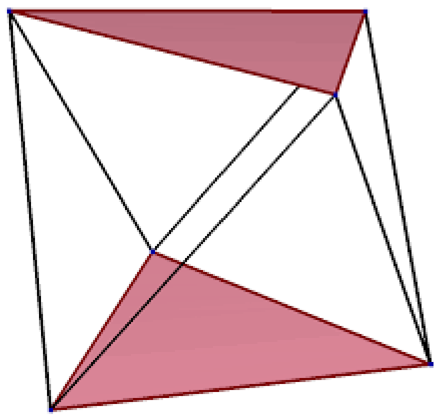


The star polygons that are highlighted above have disconnected edges, so I do not include them.

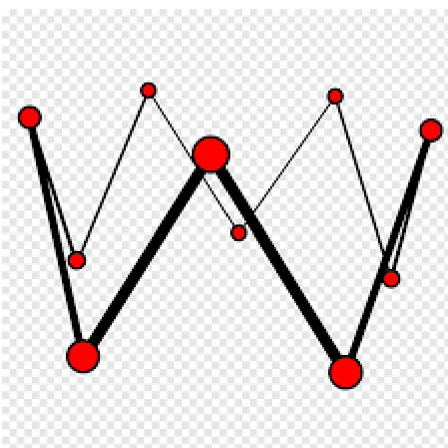
Also, polygons don't have to be two dimensional. Below are some three dimensional polygons (called skew polygons).



The polygon in the above picture has the red outline and this is called a skew square.



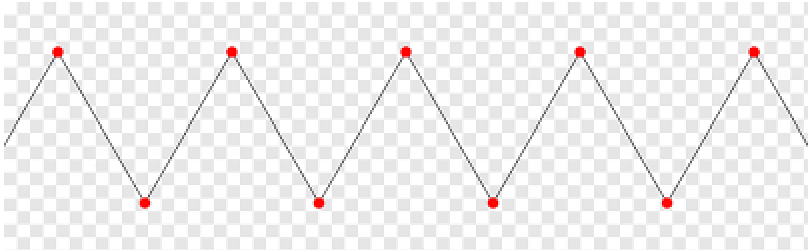
This is a skew hexagon (ignore the triangles with the colored internals).



This is a skew decagon.

The good thing about skew polygons is that they dont have internals and they also don't have area.

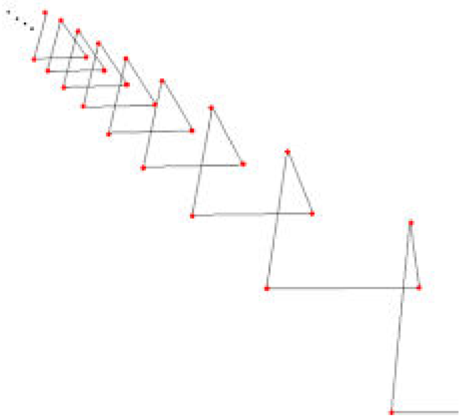
As well as a skew square, skew hexagon, skew octagon, skew decagon, there are also skew apeirogons such as these below:



This is called a zig-zag.

There are also helical regular polygons (which are regular polygons shaped like helices). They have infinitely many sides so they are also apeirogons.

This shape below is called a triangular helix.



This shape is a square helix.

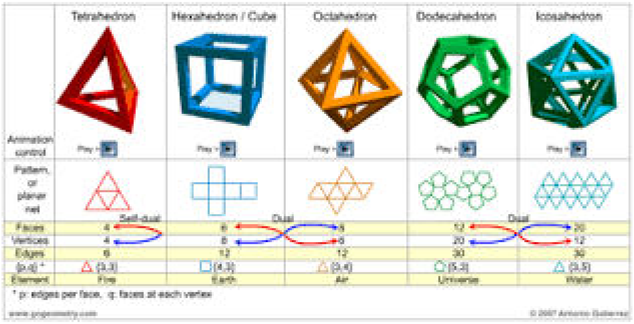
and so on.

Now, I can talk about polyhedra.

I assume that you will already know what a polyhedron is.

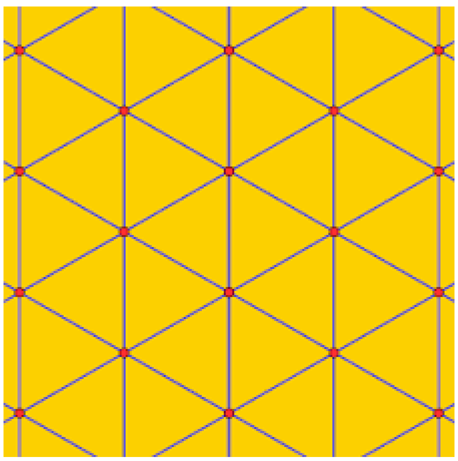
My definition of a regular polyhedron is a polyhedron that is vertex transitive,edge transitive and face transitive.

Face transitivity is like edge transitivity but with the faces on a polyhedron. If you look up regular polyhedra on YouTube, you will mostly find videos proving why there are only five regular polyhedra. I advise you to watch one of those videos before reading this. Below is what they call the five regular polyhedra.

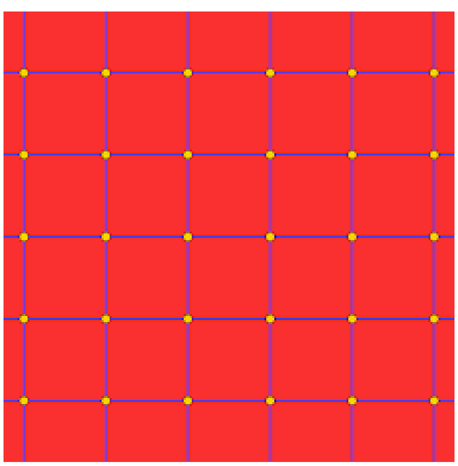


But they assume that a regular polyhedron can't have infinitely many sides. If you are smart and believe that polyhedra CAN have infinitely many sides, then you will also have these shapes below.

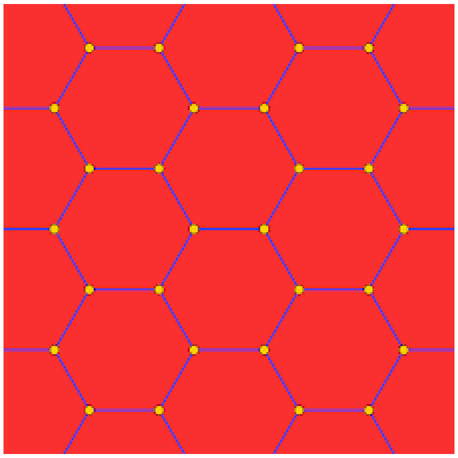
The triangular tiling



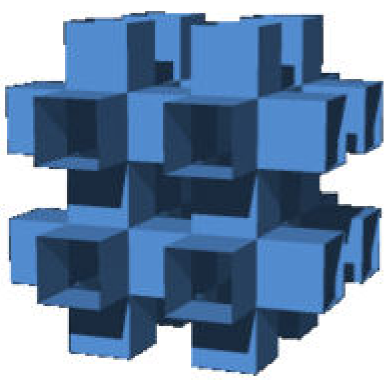
The square tiling



The hexagonal tiling

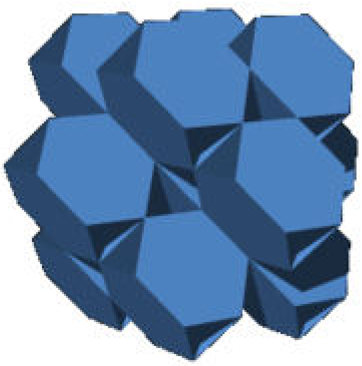


The mucube (this pattern continues on forever)



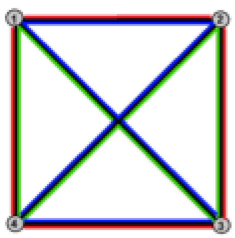
The muoctahedron (this pattern continues on forever)

The mutetrahedron (this pattern continues on forever)

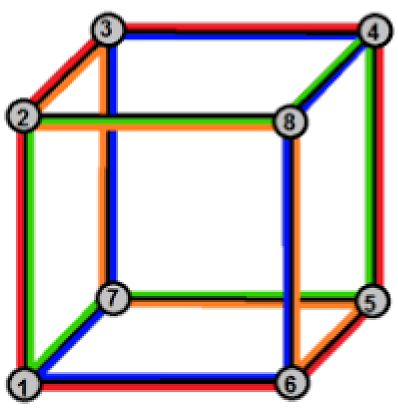


Also, if you allow skew polygons as faces, you will also have these shapes.

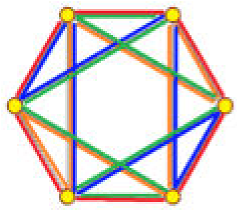
This is the petrial of the tetrahedron. It looks like a tetrahedron, but its faces are actually skew polygons, which are highlighted in the different colours to distinguish them. The petrial of a polyhedron has the edge arrangement of that polyhedron, but its faces are skew polygons which zig-zag around the middle of a shape.



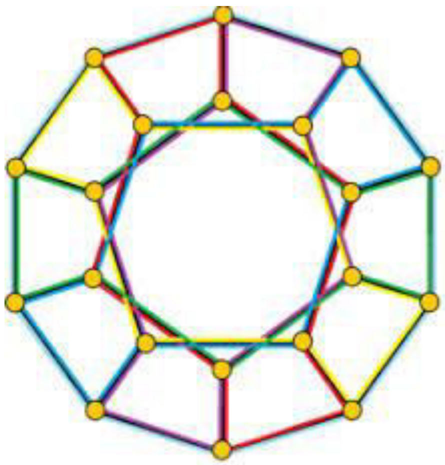
The petrial cube with numbered vertices.



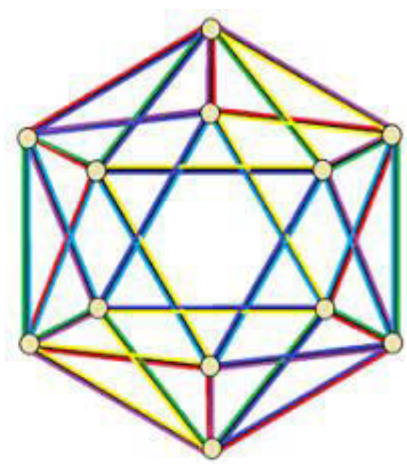
The petrial octahedron



The petrial dodecahedron



The petrial icosahedron

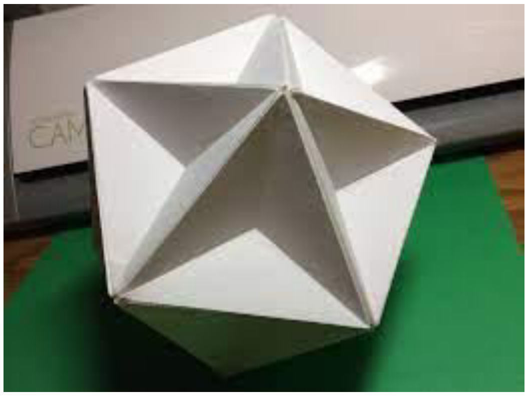


There are petrials of every polyhedron that I have mentioned in this essay and I don't have enough space to list them all.   I have listed for you the petrials of the five platonic solids.

If you allow the star polyhedra such as pentagrams, you also have these beautiful shapes called the Kepler-Poinsot polyhedra.

Great docdecahedron (made out of paper)

It has pentagonal faces that intersect eachother.

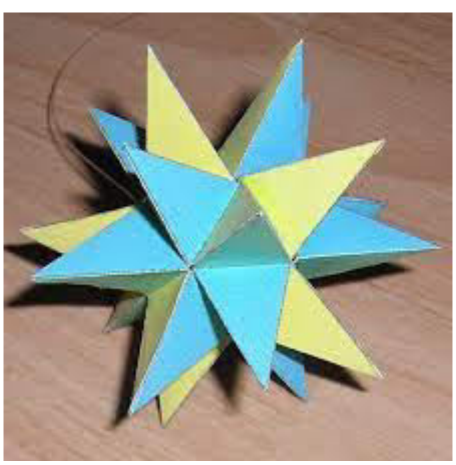


Small stellated dodecahedron (made out of paper)

It has pentagramic faces (five to a vertex).

Great stellated dodecahedron (made out of paper - of course!)

It has pentagramal faces (three to a vertex).



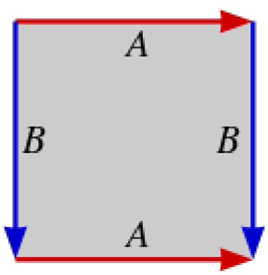
Great Icosahedran (you know what I'm going to say....made out of paper)

It has triangular faces and they intersect eachother.

Now we will move on to higher dimensional polyhedra.

Toroidal regular polyhedra

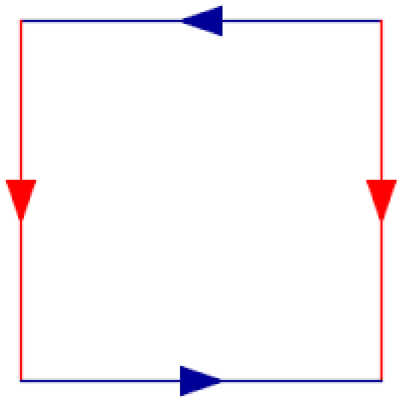
This is the fundamental polygon of a torus.



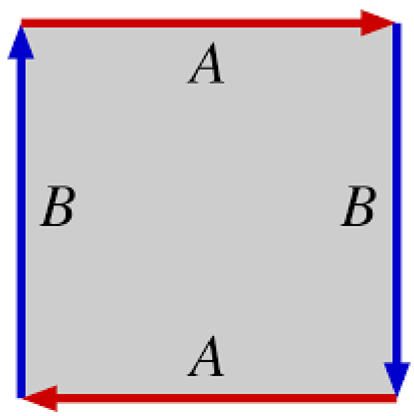
If you glue the sides together so that the colours and the arrows match, you will get a torus. If you do this with paper however, because of the difference in curvature between a piece of paper and a torus, when you try this with paper, the paper will just crinkle up.

Imagine tiling the square with squares (you can also use a rectangle as the fundamental polygon of a torus), then do the glueing. You might think that this will create a new regular polyhedron, but in three dimensions it's not possible to keep all the distances the same, however in higher dimensional space it IS possible to make a torus while preserving all the squares, which is why the title of this essay says higher dimensional space. You can also tile the higher dimensional torus with triangles or hexagons and you have more regular polyhedra. When you include these toroidal polyhedra, you have infinitely many regular polyhedra, because you can tile the rectangle with any number of squares you want as long as the number of squares is bigger than a certain number that I haven't calculated yet.

You can also do the same thing with the fundamental polygon of a klein bottle (picture below), and end up with klein bottle polyhedra.



Or do it with the fundamental polygon of a real projective plane (picture below).



Now, I will talk about some regular polyhedra in 3D space, that don't have names, but YouTuber Jan Misali came up with names for them which I will use.

If you take a square tiling, turn all the squares into square helices (some clockwise and some anti-clockwise in a symmetrical way) and keep them connected, you have a helical square tiling. If you take a triangular tiling, turn all the triangles into triangular helices (clockwise and anti-clockwise in a symmetrical way) and keep those connected, you have the helical triangular tiling and it is the same thing with the hexagonal tiling. You can take the regular tilings as well as their petrials and raise some of the vertices higher than the others and you get the blended square tiling. If you do the same thing with the triangular tiling, you get the blended triangular tiling. Do the same thing with the hexagonal tiling and you get the blended hexagonal tiling. It is the same thing with their petrials.

I would just like to mention another operation that you can do to shapes that you may have already heard of called the dual. For example, the dual of the cube is the octahedron because if you take a cube and place in the middle of every face a vertex and if two faces of the cube are connected by an edge, then connect the corresponding vertices by an edge. Keep doing that as much as you can, then remove the cube - the shape you end up with is the dual of the cube. I know this is a bit complicated - it is only so because I can't use animation. If I could use animation to show you, it would be much simpler to explain.

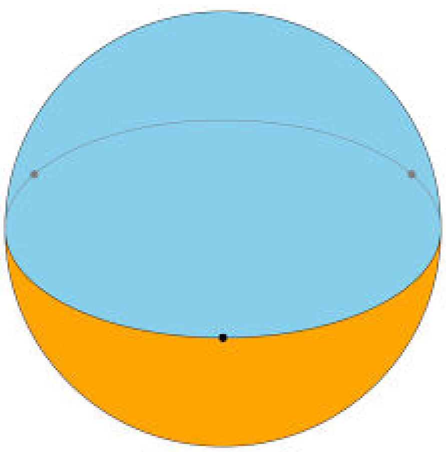
Every polygon and polyhedron has a dual. The tetrahedron and some other shapes are self-dual, meaning that they are their own dual.

If you take a mucube, make diagonal cuts on its faces, one cut per face and do it in a symmetrical way where there are six edges to a vertex, keep all the cut lines and remove the original mucube, you end up with a halved mucube. The halved mucube has hexagonal faces. It also has a petrial. The petrial halved mucube has a dual.

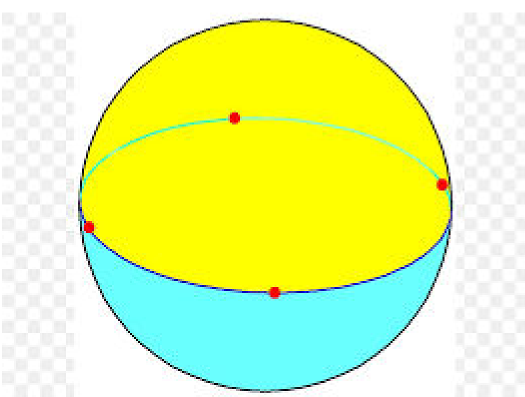
If you take a muoctahedron, remove half of the vertices from each hexagon and then look at it closely you will see some infinite stacks of triangles. Turn each stack of triangles into a triangular helix - you now have a skew muoctahedron.

Remember the platonic solids with triangular faces - icosahedron (five triangles to a vertex), octahedron (four triangles to a vertex), tetrahedron (three triangles to a vertex)...what comes next? Well, it should be a shape with two triangles around each vertex. This results in two triangles stuck back to back - it is called a triangular dihedron. If you do the same thing with squares, you get a square dihedron and you can probably guess what it's called when you do the same thing with pentagons - a pentagonal dihedron. A dihedron is a polyhedron with two faces. Below are some pictures of spherical (rounded) dihedra.

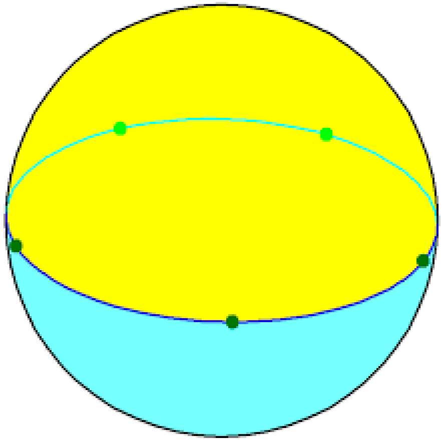
The Triangular Dihedron



The Square Dihedron



Pentagonal Dihedron

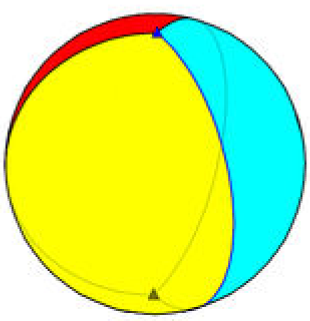


and so on...

There is also an apeirogonal dihedron. It is also a tiling of the plane. (picture below)

The dihedra have duals called hosohedra.   Below are some pictures of spherical (rounded) hosohedra.

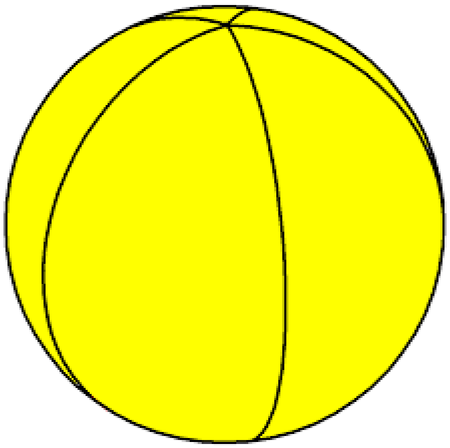
The Triangular Hosohedron (the triangular dihedrons dual)



It has digonal faces (three to a vertex).

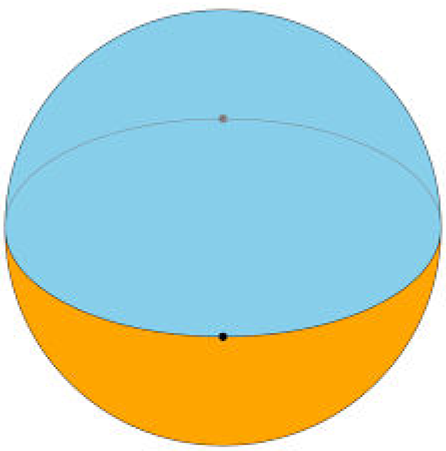
I could not find a picture of the square hosahedron (the square dihedrons dual).

The Pentagonal Hosohedron (the pentagonal dihedrons dual)

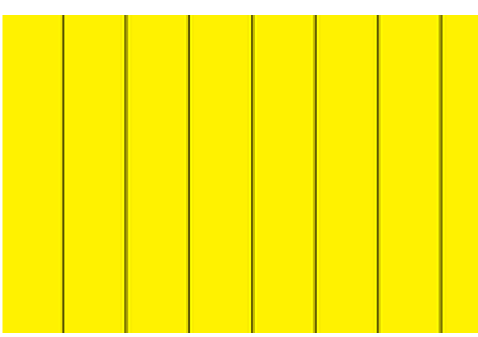


and so on...

There is also a digonal dihedron, which is self-dual, and so it is also the digonal hosohedron. Below is a spherical (rounded) digonal dihedron.



There is also an apeirogonal hosohedron (the apeirogonal dihedrons dual) - picture below.

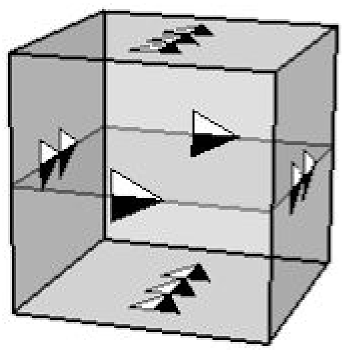


It is also a tiling of the plane. The vertices are at infinity.

Now I will mention some more higher dimensional regular polyhedra.

Remember the fundamental polygon for a torus? - there is a three-dimensional equivalent of that. If you glue it together by matching up the colours and arrows, you will get the four dimensional equivalent of a torus - the three-torus. It can exist in higher-dimensional space. If you get a cuboid like piece of a mucube or muoctahedron (like the images I've shown earlier), place it inside the cuboid and do the glueing, matching up the arrows and colours, you will have more regular polyhedra (you can also the same thing with all of these other cuboid shaped fundamental polyhedra. Fundamental polyhedra are just the three-dimensional equivalent of fundamental polygons).

Below is a picture of the three-dimensional analog of the fundamental polygon of the torus (instead of colours, it has different types of arrows).



Take a square tiling, seperate all the squares, turn each square into a square helix (vertical), connect the square helices with some more horizontal square helices (the distance between the centres of two previously connected edges is the same as the edge length of the squares). You end up with a trihelical square tiling. The petrial of the trihelical square tiling has triangular helices as faces. This is called the tetrahelical triangular tiling.

I have now covered all of the regular polyhedra that I know of. If anyone finds more, I would love to know. I hope that you have enjoyed learning a bit more about this topic - it is one of my favourite areas of maths and please believe me that digons, dihedra and hosohedra are legal and do not believe in discriminating against zero!