The Representation of Complex Numbers as Matrices

In this essay, I will explore how complex numbers can be represented as 2x2 matrices.

Complex numbers are represented as $a + bi$ where *a* and *b* are real numbers, and $i = \sqrt{-1}$. All complex numbers can be represented on a complex plane, known as an argand plane, just how all real numbers can be represented on a number line. All complex numbers can be added, multiplied and reciprocated, and contain a real and imaginary part. For example, if $z = 3 + 4i$, then $Re(z) = 3, Im(z) = 4$.

Matrices are a set of numbers arranged in rows and columns to represent data in a rectangular array. A matrix is in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Matrices can too be added, multiplied and reciprocated. The 2x2 identity matrix, written as I_{2x2} or 1_{2x2} (in blackboard bold), is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. A matrix has a determinant, $det\, A = ad - bc$, an inverse, $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and a transpose, $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

As previously discussed, the 2x2 identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This matrix can be used to represent the number 1, and they share many properties. For example, when you multiply matrix *A* by A^{-1}, it always equals the identity matrix. Also, when you multiply matrix *A* by the identity matrix (or the other way round), it equals matrix *A*. Using the identity matrix, it is possible to represent every real number as a matrix, where $x \equiv \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\ x \in \mathbb{R}$

Recently, I asked the question, "If there is a matrix to represent 1, could there be a matrix to represent *i* as well?" This new matrix to represent *i* would share all the properties of the number *i* and would allow us to represent all complex numbers as matrices. I did some calculations to find this matrix. Here, I call this matrix A, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ a, b, c, d \in \mathbb{R}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$A^2 = -\mathbb{1}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$a^2+bc=-1 \quad ab+bd=0$$
$$bc+d^2=-1 \quad ac+cd=0$$

Here, I used the property of *i*, where $i^2 = -1$. If this property is true for *i*, then it must be true for the matrix that represents *i*. This helped me find some information about the constants *a,b,c,d.* Now that I new some information, I needed to use another property of *i* to find matrix *A.*

Next, I used the property of *i*, where $|i| = 1$, and concluded that $\det A = 1$ This would mean matrix *A* is an orthogonal matrix. Therefore, $AA^T = 1$. After some calculations, I found that $a = 0$, $d = 0$ and $b = -c$.

$$AA^T = \mathbb{1}$$
$$AAA^T = -\mathbb{1}A^T$$
$$A = -A^T$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix}$$
$$a=-a \Rightarrow a=0$$
$$d=-d \Rightarrow d=0$$
$$b=-c$$

With all the information I had gathered, I found the values of *a,b,c,d*. However, I now had two matrices for *A*, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This poses two questions. Why are there two matrices here and which one should we use?

$a^2 + bc = -1,\ b = -c$
$b(-b) = -1$
$b = \pm 1,\ c = \mp 1$

The answer to the former question is quite simple: there are in fact two complex numbers which have the properties used to find matrix *A*, *i* and *-i*. Though I attempted to find the matrix which represents *i*, I accidentally found two matrices which represent *i* and *-i*, and I had no idea which was which!

At this point, I decided to research online, where I found that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ was the best matrix out of the two to represent *i*, as it also represents the rotational linear transformation $\frac{\pi}{2}$ radians anti-clockwise about the origin. This is the exact transformation to map the point *(1,0)* to *(0,1),* which are the numbers *1* and *i* on the argand plane. The other matrix I found, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, can be used to represent *-i.* **Credit: qncubed3 on YouTube for help here**

Now we have found our matrix to represent I, we can finally represent any complex number as a matrix:

$$z \equiv \operatorname{Re}(z)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \operatorname{Im}(z)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ z \in \mathbb{C} \quad \text{or} \quad z \equiv \begin{pmatrix} \operatorname{Re}(z) & -\operatorname{Im}(z) \\ \operatorname{Im}(z) & \operatorname{Re}(z) \end{pmatrix}\ z \in \mathbb{C}$$

Upon further investigation, any matrix which represents a complex number by the above formula, also represents the linear transformation mapping *(1,0)* to $(\operatorname{Re}(z), \operatorname{Im}(z))$, or *1* to *z* on the argand plane. This is useful as it allows us to find a linear transformation to map the point *(1,0)* onto any other point on a plane.

We can also derive the relationship between complex numbers and their matrix counterparts by considering the matrix which represents the anticlockwise rotation about the origin angle θ, $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

This can be written as $R_\theta = \cos\theta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\theta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and when we convert the two matrices into *1* and *i*, we get $R_\theta = \cos\theta + i\sin\theta$. This is very similar to Euler's Formula, $e^{i\theta} = \cos^\theta + i\sin\theta$. Here, we can see that multiplying by $e^{i\theta}$ will rotate our complex number θ radians anti-clockwise about the origin, and so does our matrix as a linear transformation. Writing a complex number in the form $re^{i\theta}$ allows us to write our matrix as $\begin{pmatrix} \mathrm{r}\cos\theta & -\mathrm{r}\sin\theta \\ \mathrm{r}\sin\theta & \mathrm{r}\cos\theta \end{pmatrix}$ which is equivalent to our general matrix, $a + bi \equiv \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Now we can see that a matrix in the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ represents the transformation of the rotation of some angle θ followed by the enlargement scale factor *r.* **Credit: nagwa.com for this observation.**

The matrices formed using the formula above also have all the properties of complex numbers. They can be added and multiplied to get the same results. Some examples below:

$$(2+3i)+(5-2i) \qquad\qquad (1+2i)(-3+4i)$$

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ -2 & 5 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -1 \\ 1 & 7 \end{pmatrix} \Longrightarrow 7+i \qquad\qquad \begin{pmatrix} -11 & 2 \\ -2 & -11 \end{pmatrix} \Longrightarrow -11-2i$$

Although it may be quicker to add or multiply without converting the complex numbers into matrices, it is still interesting to see that the properties of complex numbers remain after converting them. It also proves that converting them can be a legitimate method to solving problems involving complex numbers.

These matrices also share all other mathematical properties of complex numbers. This is shown by German Mathematician Jens Fehlau on his series ‘Complex Numbers, but Different’. In this series, he explores general operations, complex conjugates, powers of i, vectorspace isomorphisms and many more ideas. **Credit: Flammable Maths on YouTube**

There are a few interesting links betwwen the features of complex numbers and their matrices. If matrix A represents complex number z, then $det\, A = |z|^2$. Additionally, the transpose of any matrix representing a complex number represents the complex conjugate of that number.

Where representing complex numbers as matrices becomes very useful, is when we must find a reciporacal. Converting to a matrix is great, as it allows us to find the inverse of the matrix formed, then convert back to a complex number afterwards:

$$(3+5i)^{-1} \qquad\qquad (-2+i)^{-1}$$

$$\begin{pmatrix} 3 & -5 \\ 5 & 3 \end{pmatrix}^{-1} \qquad\qquad \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}^{-1}$$

$$= \frac{1}{9--25}\begin{pmatrix} 3 & 5 \\ -5 & 3 \end{pmatrix} \qquad\qquad = \frac{1}{4--1}\begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{34} & \frac{5}{34} \\ -\frac{5}{34} & \frac{3}{34} \end{pmatrix} \Longrightarrow \frac{3}{34} - \frac{5}{34}i \qquad\qquad = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{pmatrix} \Longrightarrow -\frac{2}{5} - \frac{1}{5}i$$

This idea can also be used to divide two complex numbers, as dividing on number by a another is the same as multiplying by its reciprocal. Here are some examples below:

$$\frac{-1+i}{5-2i} \Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ -2 & 5 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\frac{1}{25--4}\begin{pmatrix} 5 & -2 \\ 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{5}{29} & -\frac{2}{29} \\ \frac{2}{29} & \frac{5}{29} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{7}{29} & -\frac{3}{29} \\ \frac{3}{29} & -\frac{7}{29} \end{pmatrix} \Rightarrow -\frac{7}{29} + \frac{3}{29}i$$

$$\frac{2-3i}{4+7i} \Rightarrow \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 4 & -7 \\ 7 & 4 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\frac{1}{16--49}\begin{pmatrix} 4 & 7 \\ -7 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} \frac{4}{65} & \frac{7}{65} \\ -\frac{7}{65} & \frac{4}{65} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & -\frac{1}{5} \end{pmatrix} \Rightarrow -\frac{1}{5} - \frac{2}{5}i$$

This new method of finding reciporcals, and dividing complex numbers is faster and easier than the usual method of rationalising the denominator.

As you can see, representing complex numbers as matrices is possible. We have found a new matrix to represent i, and have used this, along with the identity matrix to represent any complex number as a unique matrix. As shown above, this maintains all of the properties of complex numbers, and can even be used as a quicker method for finding the reciporcals and dividing complex numbers.