The Derivation of the Metric Tensor in Einstein's Field Equations

Joshua James-Lee

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1 Introduction

The universe is massive. Indeed, there is a lot of mass in the universe, and we want to know how it all behaves, particularly when concerned with gravitational fields. Welcome to the derivation of the metric tensor in Einstein's field equations. The derivations I have done and the theory driving them have been ascertained from a virtual lecture [Eag13], and what you are about to read is the beginning of the derivation of a crucial part of Einstein's theory of general relativity, some of the most cosmically significant mathematics ever discovered and a brilliant example of applied mathematics in physics, working from the most fundamental of principles.

2 Gravity, Equivalence, Light and Marbles

Imagine you're locked within a dark box, with no ability to observe your surroundings or environment. All you are able to do is feel the forces affecting you. Intuitively you'll have imagined it on earth, where you feel the $1g$ of earth's gravity weighing you down. Now imagine the box is flying through space, accelerating at exactly $1g$. To you, an observer placed within this box, the two sensations are identical. The effect of the earth's gravitational field upon you is identical to that of the accelerating box through space. This is the Principle of Equivalence, a concept whereby due to your frame of reference, the effects caused by a gravitational field can be compared to those that are caused by the acceleration of the system of observation.

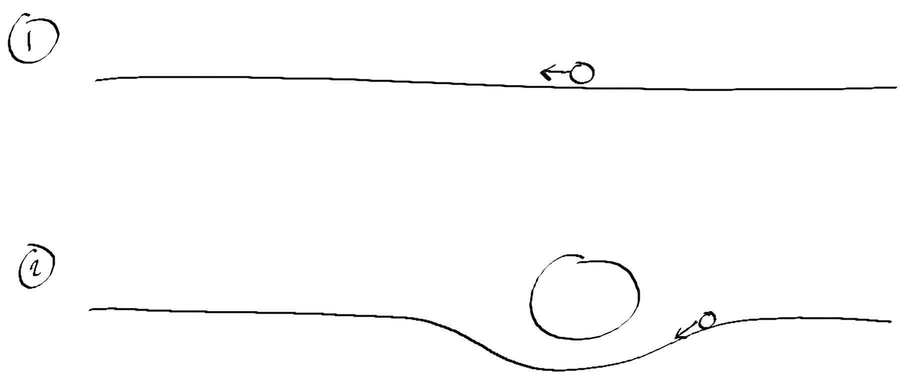
Let us now consider a beam of light that shines from one side of the box to the other. It would be reasonable to assume that the light shines as one consistent beam across the box, however this is not the case. Let the box travel incredibly fast, then faster. Imagine now a single photon traversing the container, it moves with a speed of c metres per second. As the particle of light moves, so does the box. Therefore to an external observer, the beam of light appears to bend. Using what we have just established about the equivalence between a gravitational field and an accelerating system, it can be therefore stated that light bends in a gravitational field[1].

$$F_{gravitational} = \frac{G \times m_1 \times m_2}{r^2}$$

This poses a problem. We now know that light is affected by a gravitational field, and the above equation provides us with the gravitational force between two bodies of mass m_1 and m_2 respectively. However light does not have any mass, and yet is still affected by a gravitational field. This provides an interesting paradox and one that Einstein has a solution to.

Instead of considering gravity in the traditional sense, where bodies of lesser mass fall towards bodies of greater mass, lets assert that all forms of motion can be represented by movement in curved spacetime[2].

Figure 1: Gravity and curved spacetime



In the above illustration, let us consider a marble lying on a flat plane and moving in a straight line. In the first example the path of the marble will be unaffected as there are no other bodies to interfere with its motion. However now consider the second example, where a body of some very large mass curves spacetime. The marble feels no attractive force towards the more massive body and just continues in a straight line, but to us, the observer, we can see that the marble will travel along a different path, following the curve. This is how we can explain gravity in such a way that permits light to be affected by more massive objects, through the manipulation of the fabric of the universe itself.

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[1] Proven by the Eddington Experiment. A series of observations collected during total solar eclipse of May 1919

[2] This is a daunting word to encounter on the first page of an essay. But this concept essentially condenses the three dimensions of space and time into one model. This will become less confusing as we progress. I consider a large part of my task in writing this to try and explain the concepts as simply as possible such that they may be widely understood.

3 The Field Equations

Einstein's field equations relate the geometry of spacetime, most notably its curvature, to the distribution of matter within it [Ein16].

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

The field equations can be written as they are in the form above, however they can be expanded out to show more of their components:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Where $R_{\mu\nu}$ denotes the Ricci Curvature Tensor, $g_{\mu\nu}$ the Metric Tensor, R the Curvature Scalar, Λ the Cosmological constant and $T_{\mu\nu}$ the Energy Momentum Tensor. These terms all have fairly frightening names, but we will only be focusing on one for this paper, the Metric Tensor. In total there are 10 field equations[3].

Everything on the left of the equation can be said to relate to the Curvature and Geometry of Spacetime, and everything on the right relates to Mass and Energy. In essence therefore it is possible to say that mass tells spacetime how to curve and curved spacetime tells mass how to move.

4 Deriving the Metric Tensor

Where the maths really begins[4]

4.1 Fields and bumpy bits

Consider a field. One with grass and rabbits and various other field things. This field is not perfectly level. There are bumps, and mounds and ridges all across it and you're stood on top of a particularly large bump somewhere near the centre. We're going to define the value of a field (ϕ) as the height above sea level for that tiny bit of field (essentially how far up you are at that point). What we are interested in is how ϕ changes with motion along the field, and so we now begin to consider rates of change.

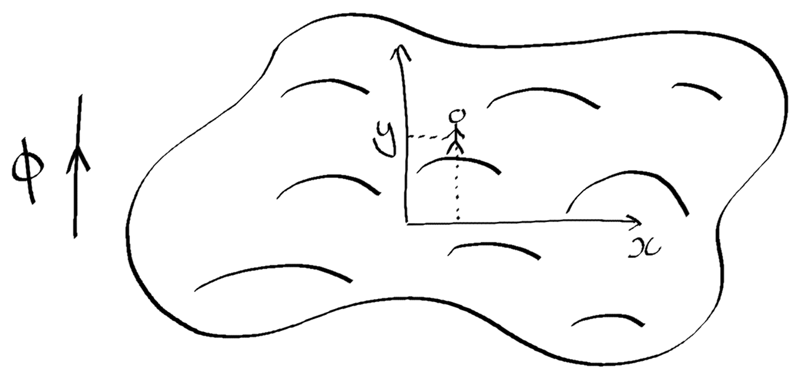
$$d\phi = \frac{d\phi}{dx} dx$$

[3] This is due to μ and ν both representing dimensions of spacetime and therefore are able to take values between 0 and 3 (inclusive) (This is a different form of notation that I'll introduce slightly later). Therefore we are left with 16 equations, where 6 of them happen to be duplicates, hence we are left with 10.

[4] *The process of this derivation is deceptively tricky and can catch you out a few pages in. So I'd recommend obtaining a substantial mug of coffee, a pencil to annotate as we go, and let some Grieg bubble along in the background because this is where things start to get interesting.*

It can be therefore said that the change of the value of the field is equal to the product of the gradient at that point in the field and the change in position along the x axis. The direction of motion is also important. If you were to find yourself on a ridge for instance, then moving in one direction might take you down a slope, but movement in another may keep you at the same height as you walk along the top of the ridge. The solution is of course, vectors.

Figure 2: A field being observed from one frame of reference.



$$d\phi_x = \frac{d\phi}{dx}dx$$

$$d\phi_y = \frac{d\phi}{dx}dx$$

You now walk not just along the x or y axis but diagonally across the field. This leaves us with a combination of changes in position, dx and dy. We can get the resultant change by using Pythagoras' theorem:

$$ds^2 = dx^2 + dy^2$$

$$d\mathbf{s} = d\mathbf{x} + d\mathbf{y}$$

$$d\phi_s = d\phi_x + d\phi_y$$

We now substitute our expressions for $d\phi_x$ and $d\phi_y$:

$$d\phi_s = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$

And we are left with an expression for the change in value of a field given movement along the x and y axis. The symbol ∂ denotes a partial derivative. A partial derivative of a function containing multiple variables is the derivative with respect to only of the variables.

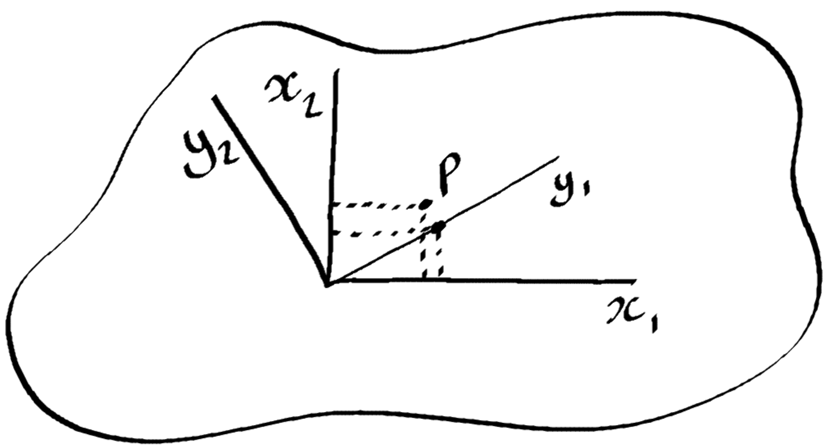
We are now going to introduce a new method of notating positions within space. x is now replaced by x^1, y is now x^2 and z is x^3. Our traditional 3 dimensional coordinate system now being represented by x^1, x^2, x^3. This has been done so that more dimensions can be added to our scope of calculations. This won't be something we worry about too much, however now we can re-write our above equation as:

$$d\phi = \sum_n \frac{\partial \phi}{\partial x^n} dx^n$$

4.2 Frames of reference

We now begin to encounter the problem of frames of reference. For something to be universally true, it must be true in all reference frames. So lets consider the field once again:

Figure 3: Field with multiple frames of reference



Now we can see different reference frames laid out on the field, and the issue at hand becomes clear. To an observer in the y frame of reference, the point **P** is in a different position to that observed by an observer in the x frame of reference. The question therefore is how, given knowledge of the gradients in the x reference frame, can we find the gradients in the y reference frame? The answer lies in the chain rule.

$$\frac{\partial \phi}{\partial y^1} = \frac{\partial \phi}{\partial x^1}\frac{\partial x^1}{\partial y^1} + \frac{\partial \phi}{\partial x^2}\frac{\partial x^2}{\partial y^1}$$

The above can be simplified further using the summation term m:

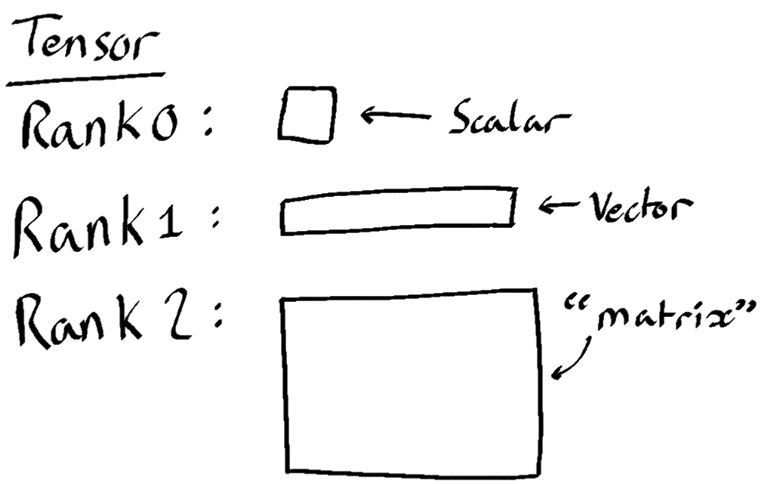
$$\frac{\partial \phi}{\partial y^n} = \sum_m \frac{\partial \phi}{\partial x^m} \frac{\partial x^m}{\partial y^n}$$

We now have an expression for the change in the value of the field with respect to ∂y^n, where n can take the value of 1 or 2. However to get one value for y, you must sum over all the x terms in the equation through the summation term m. Simply put, if we chose n to be 1, in order to find the rate of change of ϕ with respect to y^1, we'd have to sum for each value of m[5].

4.3 Tensors, Transformations and Triangles

Tensors express relationship between Vectors. A Scalar can be considered as a very fundamental form of tensor, described as a 'Tensor of Rank 0", only having the property of magnitude. Vectors, possessing both magnitude and direction are hence described as a 'Tensor of Rank 1".

Figure 4: Visual representation of Tensors



Lets refer back to one of our earlier equations, derived in 4.1, and consider it in the context of vectors:

$$d\phi = \sum_n \frac{\partial \phi}{\partial x^n} dx^n$$

[5]Realistically we could be considering x^3 and y^3 as well. However for the simplicity of the explanation, at the moment we are only considering two dimensions.

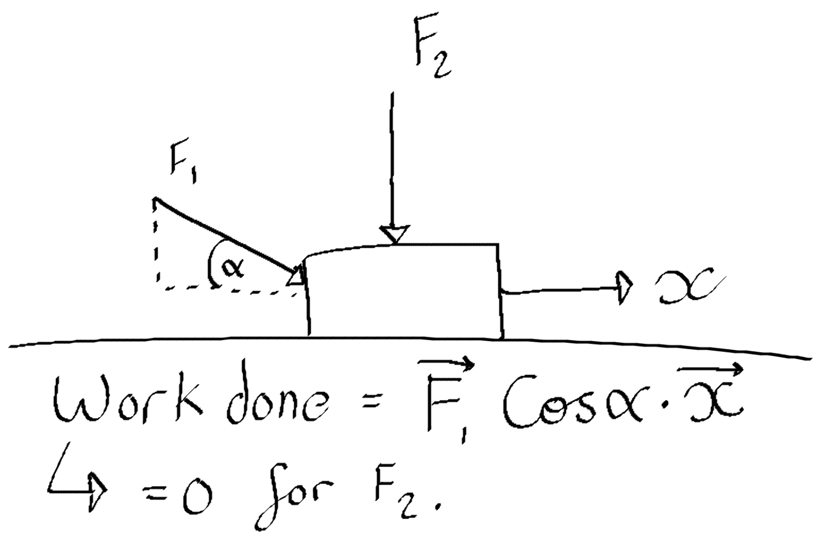
We can re-write this as[6]:

$$V_y^n = \sum_m \frac{\partial y^n}{\partial x^m} V_x^m$$

V_y^n represents a vector within the y frame of reference with coordinate n, m is once again the summation term and V_x^m is a vector in the x frame of reference.[7]

Returning to Tensors, a tensor of rank 2 can be considered as a combination of Vectors where there is a definite relationship between the two.

Figure 5: Example of a relationship between two vectors: F and x



Considering the example above (Figure 5) We can consider the Tensor to be the relationship between vectors F and x. When we are working with F_1, the force has a horizontal component and so x has a magnitude greater than zero. However if we regard F_2, then there is no horizontal force and so the work done is equal to 0, which is the value of the tensor in that circumstance. Crucially, if a Tensor has a value of 0 in on frame of reference, it must have a a value of 0 in all other frames of reference.

[6]It is important to recall the new coordinate system being used, as defined in 4.1. x and y are not individual coordinates but rather they are individual frames of reference with coordinates of the following format: (x^1, x^2).

[7]Essentially we're now transitioning into considering Vectors. Understanding the steps to get there is not vital but its helpful to know that we're now working with them.

Given the above, particularly that a tensor is a combination of vectors, a tensor can be described as follows, where m and n are dimensions of space:

$$T^{mn} = A^m B^n$$

Now building off the equation for V_y^n we've just obtained, we can establish that:

$$A_y^m B_y^n = \sum_r \frac{\partial y^m}{\partial x^r} A_x^r \sum_s \frac{\partial y^n}{\partial x^s} B_x^s$$

We now know that both $A_y^m B_y^n$ and $A_x^r B_x^s$ are tensors, so we can re-write the equation once again and simplify it as:

$$T_y^{mn} = \sum_{rs} \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} T_x^{rs}$$

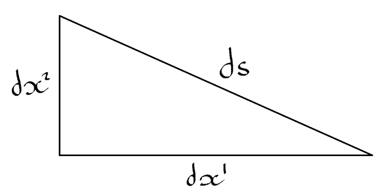
The above is known as the Contravariant Transformation and is a tensor transformation between two frames of reference, x and y. This then give us the Covarient Transformation, which will prove helpful later on. The key difference between the two transformation is that the the covarient transformation is composed of $\frac{\partial x}{\partial y}$ terms instead.

$$T_{mn} = \sum_{rs} \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} T_{rs}$$

Where T_{mn} is in the y frame of reference and T_{rs} is in the x frame of reference.

4.4 Pythagoras and Kronecker walk into a bar

Now we return to Vectors. Lets picture a right angled triangle and give the two shorter sides lengths of dx^1 and dx^2 respectively. The hypotenuse we'll call ds, and we can therefore express ds^2 as follows:



$$ds^2 = dx^{1^2} + dx^{2^2}$$

However we encounter an issue, we want to be able to express ds^2 in such a way that we can account for all dimensions of space. Using a summation yields the following:

$$ds^2 = \sum_m dx^m dx^m$$

This works for the single summation term m, but not for two summation terms, m and n, as there would be instances where $m \neq n$. We need to avoid having $dx^{1^2} dx^{2^2}$ terms for our equation to work, hence we introduce the Kronecker delta (δ_{mn}), a function that takes the value of 1 if $m = n$ and 0 if $m \neq n$.

$$ds^2 = \delta_{mn} \sum_{mn} dx^m dx^n$$

Now, re-writing our equation from 4.1:

$$d\phi = \sum_n \frac{\partial \phi}{\partial x^n} dx^n$$

as:

$$dx^m = \frac{\partial x^m}{\partial y^m} dy^r$$

Where the summation term r is implied. We can now substitute this into our expression for ds^2 and rearrange it, leaving us with:

$$\partial s^2 = \delta_{mn} \sum \frac{\partial x^m}{\partial y^r} \frac{\partial x^n}{\partial y^s} \cdot \partial y^r \partial y^s$$

A full expression for ∂s using both ∂x and ∂y terms and the Kronecker delta, where the first part of the product is known as the *Metric Tensor*.

$$g_{mn} = \delta_{mn} \sum \frac{\partial x^m}{\partial y^r} \frac{\partial x^n}{\partial y^s}$$

4.5 The importance of the metric tensor

An interesting observation to be made lies in the similarity of two of the equations we have just derived:

$$ds^2 = \delta_{mn} \sum_{mn} dx^m dx^n$$

$$\partial s^2 = g_{mn} \cdot \partial y^r \partial y^s$$

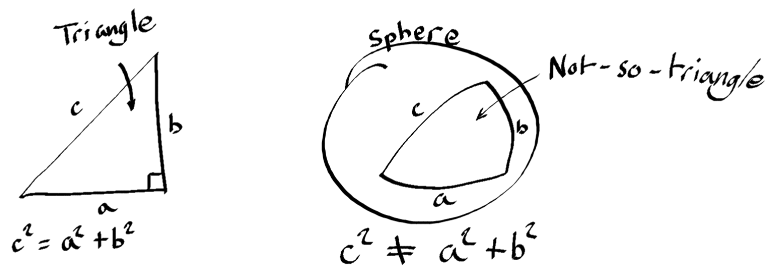
It should leap out at us that what we've just established as being the metric tensor[8], occupies a similar role as the Kronecker delta. Indeed, for flat space,

[8] Remembering that the metric tensor contains a summation term.

g_{mn} will reduce to δ_{mn} and so for flat space the metric tensor will equal 1 or 0 depending on if $r = s$ or not.

However the importance of the metric tensor becomes evident when considering curved space[9]. Pythagoras' theorem works in flat space, but were we to consider the surface of a sphere, then the sides of Pythagoras' triangle would be curved and so our formula for $ds^2 = dx^{1^2} + dx^{2^2}$ would not work. This is what the metric tensor corrects for. g_{mn} can therefore be considered as a device that makes corrections to Pythagoras' theorem when working with a curved space.

Figure 6: Why the metric tensor is important



5 The end...

... of this bit[10]. There is really a lot more to these equations, and that's only their derivation, let alone their application. These are equations that have profound relevance in modern physics, capable of modelling gravitational phenomena such as rotating black holes or aiding in the study of gravitational waves. This is a brilliant example of applied mathematics within physics. We've looked at mechanics, geometry, derivatives, algebra and more, and as I continue to battle my A-Level courses I cannot help but wonder that there must be so much more to discover and build up to being able to express out there in our universe that we might not know about today.

[9] Curves feature quite prominently in spacetime geometry.

[10] However, as I am now sat at my desk and preparing to submit this paper and staring at the 14 pages of algebra I had derived in hope of writing about, only to be thwarted by word count and deadlines. Here are the section titles of what lies ahead in terms of the derivation: *The Christoffel Symbol: ("Derivatives break everything", "Using the Christoffel Symbol and why"), The Ricci Curvature Tensor: ("Curving spacetime and confusing vectors", "Commutators", "Putting it all together"), Stress Energy Momentum Tensor: ("Geodesics, Proper Time and Newton joins the fray", "Divergence theorem, momentum and matricies"), The field equations: "Assembly required"*

References

[Eag13] B. Eagle. *Einstein Field Equations - for beginners!* 2013. URL: https://youtu.be/foRPKAKZWx8.

[Ein16] A. Einstein. "Die Grundlage der allgemeinen Relativitätstheorie". In: *Annalen der Physik* 354.7 (Jan. 1916), pp. 769–822. DOI: 10.1002/andp.19163540702.