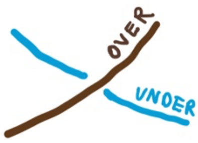
Nikita Handel

Knotty Chocolate Babka

The first time you learnt how to make a knot was probably when tying up your shoelaces. There are many ways to do this, but all of them involve taking a piece of string and tying it up into a closed loop. While this seems elementary, there is a whole area of mathematics dedicated to studying different kinds of knots: Knot Theory. Mathematicians are interested in understanding how to recognise different knots and how we can classify them.

When studying knots, it is important to have lots of pictures to help visualise what is going on. However, pieces of string are boring! We have instead chosen a slightly tastier representation of a knot: homemade chocolate babka. Babka is a traditional Jewish bread, which is rolled into a rope with layers of chocolate and shaped into various patterns. While most Jewish bakeries don’t tend to sell mathematical knots, we can learn a lot just by looking carefully at them.



We will also use knot diagrams to help us, in case the chocolate swirls become distracting. When two strings cross, the string which goes over (brown) is drawn as a line, while the string which goes under (blue) is drawn as a broken line.

The simplest knot possible is called the unknot, and it is essentially just a circle. Our babka representation has some turns in it to create a swirl, but we can easily unwind them to get a simple loop, shown by the diagram.

Next, we have the trefoil, which is made by tying a simple overhand knot (as if you were tying up your shoelaces) and connecting the two loose ends.

Our final two babka knots are slightly more fun. Before we stick the two ends of our dough ropes together, we first weave the rope in and out of itself, to create interesting patterns.

A

B

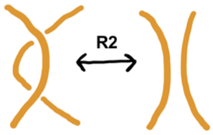
The babka representations and diagrams help us see what the knot looks like. You might wonder whether there is a unique way to draw a particular knot. If there is, this could help us when we try to classify different knots. Let’s consider an example: are these two knots the same?

We can see that knot A is the trefoil, shown above. Knot B has a different drawing to A, however if you try making it out of some string (or babka dough), you can see how it is the same knot as A. Knot B is the trefoil, but the top loop has been twisted over itself. Since we can undo this action, we're left with the same knot.

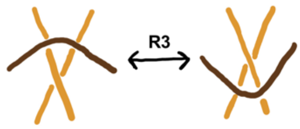
So, if we can perform an 'undoable' action on a knot, this doesn't actually change the knot we have. These actions are known as Reidemeister moves.

There are 3 types of Reidemeister moves, which we will refer to as R1, R2 and R3. All of these are easy to visualise.

R1 is twisting or untwisting part of the knot, just like how we twisted the trefoil to get knot B.



R2 moves two strings of the knot, so they either cross over each other at two points or don't cross at all.



R3 slides a string of the knot over a crossing point (of two other strings). You can see how the brown string slides from above the crossing point to below the crossing point when you apply R3.

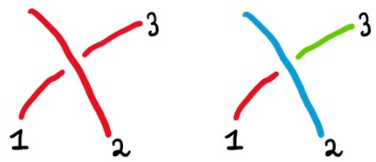
When we perform Reidemeister moves on a knot, we are simply twisting and moving parts of it, so it looks different, but is still the knot we started with. When we perform successive Reidemeister moves on the unknot, we can see how our final knot diagram looks very different to the initial one.

So, we've answered our initial question: there does not exist a unique way to draw a knot, because Reidemeister moves change how they look. This creates an issue for us: how are we meant to classify different knots, if we don't have a unique way of identifying them? What if the trefoil is simply the unknot, with a few Reidemeister twists and turns in it?

To answer this question, we need a system to identify knots, which stays the same regardless of any Reidemeister moves. Mathematicians call these systems knot invariants, and there are various kinds. We will explore whether the trefoil and the unknot are the same, through our first knot invariant: tricolourability.

When you zoom into a crossing point on a knot diagram, you can see that there are 3 parts, which we will call strands. A knot diagram is tricolourable if it satisfies two conditions:

1. We can assign each strand a colour, so that at each crossing point either all the strands are the same colour, or they are 3 distinct colours.
2. Overall, the knot diagram uses 3 distinct colours.



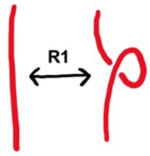
1-Colour Crossing Point   3-Colour Crossing Point

We can try colouring our babka trefoil and unknot to understand whether the baked representations are tricolourable.

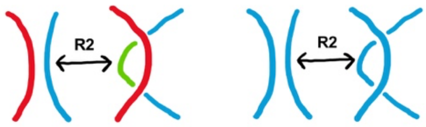
As we can see, our babka trefoil is tricolourable, as the diagram features 3 colours and has 3 colours at each crossing point. Our babka unknot is not tricolourable as the diagram only features a single colour.

What we now need to consider is whether tricolourability is actually invariant under Reidemeister moves. We can do this by simply looking at some more pictures.

R1 preserves tricolourability, as twisting a piece of string doesn't change our colouring. This is because a twist point only has 2 strands, so they must all be the same colour (as having 3 distinct colours is impossible), like the initial untwisted string.



For R2, we have two cases to check. If we have two different coloured strings, when we perform R2 we simply add a new colour (green) to preserve tricolourability at each crossing. If we have two strings which are the same colour, under R2 all strands remain that colour. So, we can see how when we perform R2 on a tricolourable section we are always left with a tricolourable section.



Finally, R3 preserves tricolourability as well. Sliding the green strand from above the crossing point to below the crossing point simply swaps the colours of the red and blue strands. This idea of 'swapping' colours works for other cases as well.

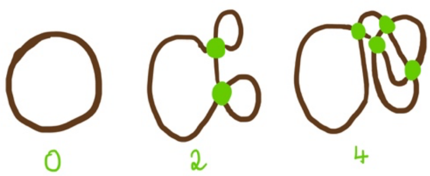
Thus, we can see that all Reidemeister moves preserve tricolourability. This is clear when we consider the unknot diagrams we produced earlier when performing Reidemeister moves: it is impossible to tricolour either of them.

We now have all the ingredients ready to prove that the trefoil is not the unknot.

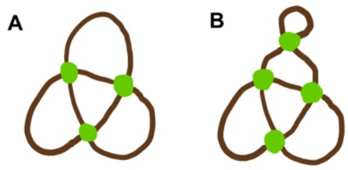
Firstly, we know that our babka representation of the unknot is not tricolourable, and our babka representation of the trefoil is tricolourable. We also know that tricolourability is invariant under Reidemeister moves. This means that all representations of the unknot are not tricolourable, and all representations of the trefoil are tricolourable. For two knots to be equivalent, they need to need to be both tricolourable, or both not tricolourable. Since the unknot is always not tricolourable, and the trefoil is always tricolourable, it follows that these knots are not the same.

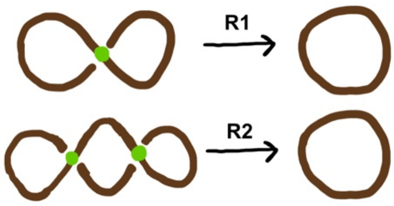
So, we've not only answered our question, but we have also mathematically proven that the trefoil is different to the unknot, through exploring the knot invariant of tricolourability. Why stop there? Let's explore another knot invariant: crossing numbers.

The crossing number of a knot is the least number of crossing points in any drawing of a knot. We can see how the crossing number varies for the different diagrams we produced for the unknot. Each crossing point is represented by a green dot. Since we have an unknot diagram with 0 green dots, this must be the



unknot's crossing number, as 0 is the least possible number of crossing points (since we can't have negative crossing points).
We can also look at our trefoil diagrams to consider how many crossing points they have. Trefoil A has 3 green dots while trefoil B has 4. Since we only care about the least number of crossing points, we know the trefoil's crossing number is at most 3. That is, it could be 0, 1, 2 or 3. We can produce infinitely many trefoil diagrams by performing successive Reidemeister moves, so how do we know what the least number of crossing points is?





Let's first take a closer look at knot diagrams with a crossing number of 1 and 2. When we draw a knot with a single crossing point, we see that this is simply the unknot with an R1 twist. Likewise, when we draw a knot with two crossing points, this is the unknot with an R2 move.

We can see how it is impossible to have a knot with a crossing number of 1 or 2, as you can always reduce these knots to a diagram with 0 crossing points: the unknot. This is helpful for us, as it narrows down the options for the trefoil's crossing number- since it can't be 1 or 2, it can only be 0 or 3.

Now, we know that the trefoil is different to the unknot, from our tricolourability proof. This means the trefoil and unknot cannot have the same crossing number, since crossing numbers are a knot invariant. Thus, it follows that the trefoil must have crossing number 3, as the unknot has crossing number 0.

We now have two invariants which differentiate between the trefoil and the unknot: tricolourability and crossing numbers. So, if we are given a trefoil knot diagram, we can recognize that it is not the unknot. What if we generalised this idea? If we are given any knot diagram, is it possible to determine whether it is simply the unknot? We can see how tricky this could be by looking at Haken's unknot (we promise we did create a babka version, however unfortunately the oven was too small to bake it). It looks incredibly complicated, but it is just an unknot with a huge number of Reidemeister moves.

HAKEN'S UNKNOT

Wolfgang Haken proved the Unknot Theorem in 1961- that is, it is always possible to determine whether a given knot diagram is equivalent to the unknot. This can be done through an algorithm. An algorithm here is a list of instructions that takes a given question (is this knot the unknot?) and answers it in a finite amount of time.

While several of these algorithms exist, in practice they are terribly unhelpful for mathematicians as they are performed in exponential time, where the length of running time grows exponentially as the algorithm inputs increase. Mathematicians are currently puzzling over the unsolved problem: can we recognize an unknot in polynomial time? Polynomials grow at a slower rate than exponentials, so the length of running time doesn't increase as much. In other words, can we create an unknotting algorithm which can be run in a shorter timeframe?

The famous geometer William Thurston once described this problem by saying, 'A lot of people have thought about this question ... but this has been a very hard question to resolve.' While the problem is still unsolved and incredibly challenging, there have been some exciting recent developments. In 2021, Marc Lackenby was able to produce an unknotting algorithm which runs in quasi-polynomial time, which is still slower than polynomial time, however it is much faster than exponential time. This means mathematicians are closer to solving the problem, and are now working to shave off extra time by making the algorithm more efficient.

So, we started off with a shoelace, some babka dough and a little bit of imagination, and we have been able to explore some exciting areas of mathematical research. Knot Theory is a rich area of mathematics which is creative, challenging and fun. We hope this exploration will prompt you to research even more. And, most importantly, bake some babka.

Bibliography

1. https://thegreatbritishbakeoff.co.uk/recipes/all/paul-hollywood-chocolate-babka/
2. Adams, C., *The Knot Book*.
3. Gilbert, N. and Porter, T., *Knots and Surfaces*.
4. http://web.math.ucsb.edu/~padraic/ucsb_2014_15/ccs_problem_solving_w2015/Tricolorability.pdf
5. https://www.maths.ox.ac.uk/node/38304
6. https://www.youtube.com/watch?v=IVU8hae2T7Y
7. https://www.youtube.com/watch?v=emKrcoW612Y