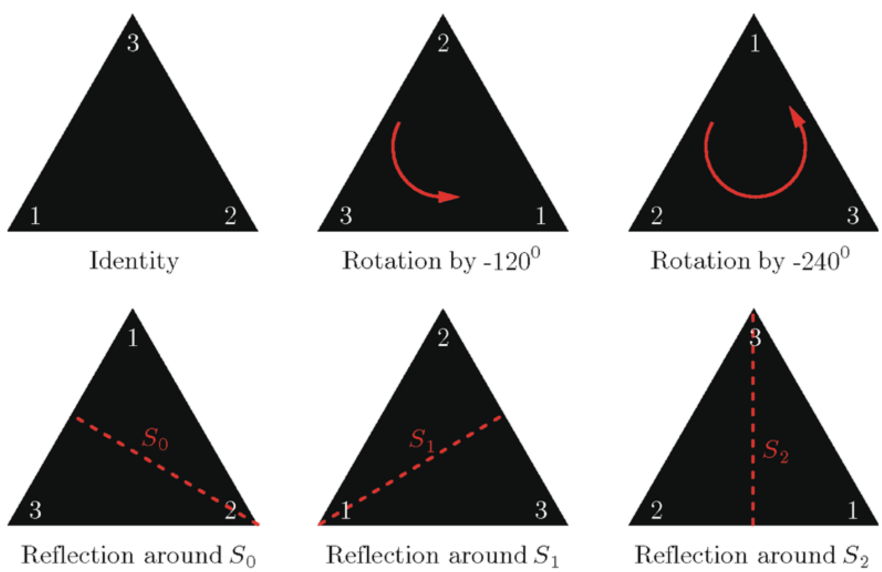
The Mathematics of Symmetry (and Monsters)

Vedant Lohia

March 2022

If one day you happened to be walking through a 196,883 dimensional space, you might come across quite a fascinating object that mathematicians till this day still don't know why exists. This of course sounds absurd and perhaps arbitrary (for example why the number 196883?) but the "existence" of such an object and its significance is a spectacle in the wonderful branch of maths known as abstract algebra, more specifically: group theory.

Most people, when hearing the word "maths", picture numbers, calculus or maybe trigonometry and these are certainly all valid and are founded very much in every day problems and reality. Group theory is a less obvious case and is a prime example of how "maths" can founded in concepts seemingly detached from numbers and our classic notion of the subject. To begin to make sense of what group theory is about, let's consider an equilateral triangle and the following question: "how many ways can I transform the shape such that it looks the same after the transformation has been applied?".



After a bit of thought we could say that there are 6 "actions" that leave the triangle looking the same. These 6 actions constitute a group and more technically what is known as the dihedral group of order 6 (D_6) with the order being the size or number of elements in the group. We have a rotation through 120° and through 240°. We also have a reflection along the three vertices. The 6th one corresponds to "doing nothing", a rotation through 0° or 360° if you will, which technically counts as a symmetry as it leaves the triangle unchanged (or "preserves its structure" if we want to be more abstract) and it is known in the field as the "identity" element. So you may now think that it's all about shapes and geometry. Well, usually in maths we tend to start with a very specific case and then abstract the concept as far as possible. Thus to say that group theory is all about rotating and reflecting shapes would be a gross injustice. But why could the symmetries of an object such as a triangle potentially be useful? It isn't so much about the triangle itself but more so how its intrinsic symmetries and elements of its "group" stretch far beyond this particular example of a shape. To demonstrate what this means, let's now consider the permutations of 3 objects in a row. With an elementary knowledge of combinatorics we can say that there are 6 or 3! permutations. The fact that the two totals amount to 6 in both cases is no coincidence and it is by the definition of a group that we can bring together two distant areas of maths (combinatorics and Euclidean geometry) into one. Through the lens of group theory, these two very distinct problems are in fact identical and we call their respective groups, that is, the symmetries of the equilateral triangle and the number of ways to permute a row of three objects, isomorphic (the same).

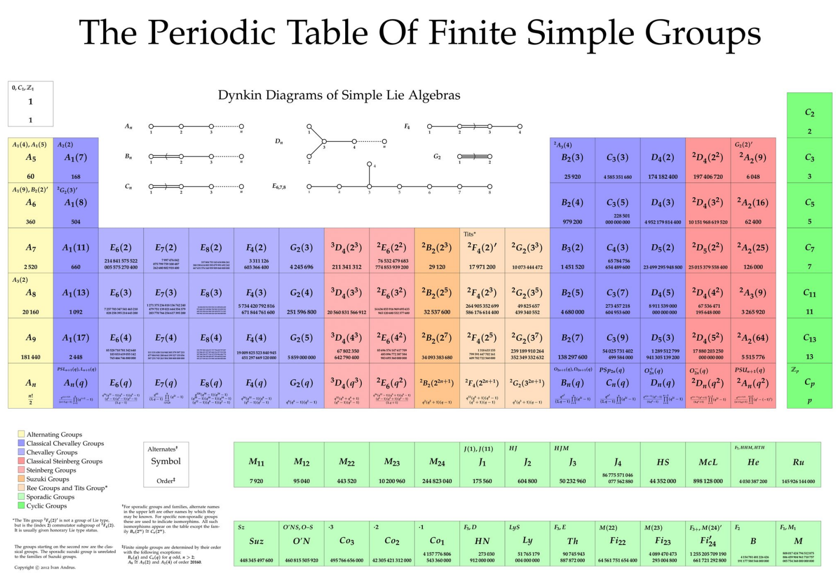
If you're still not convinced or at least intrigued, group theory has been used to solve the Rubik's cube and is required in physics to derive an incredible result known as Noether's Theorem which states that for a symmetric transformation in a physical system (i.e if I displace an object by a certain amount leaving it unchanged) there must have been a quantity that was conserved and therefore there exists a corresponding conservation law for that system (e.g the conservation of energy). So what is the formal definition of a group? A group is a set closed under a binary operation such that for all elements within the set, the binary operation is associative, there exists an identity element and there exists, corresponding to each element, an inverse element. The binary operation can be defined as anything and stretches far beyond our normal 4 operations along with the elements which don't have to be numbers at all (they could be matrices for example). It's best to mathematically write out the conditions for a group as they are easiest understood in that form:

$$a * (b * c) = (a * b) * c$$

$$e * a = a * e = a$$

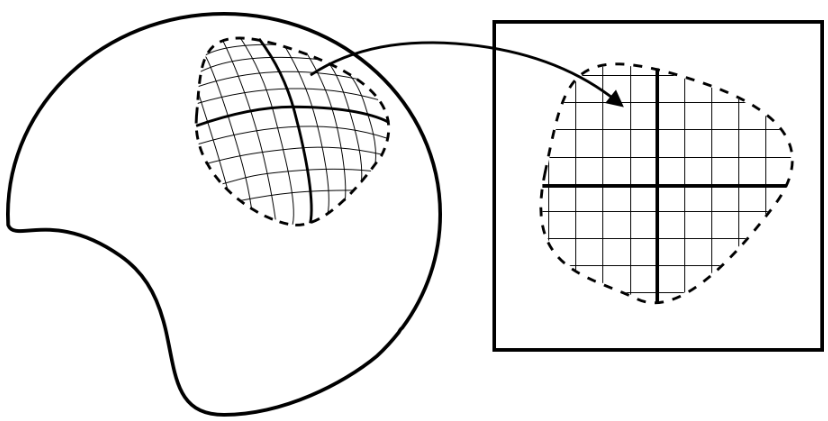
$$a * a' = a' * a = e$$

e here denotes the identity element, a' denotes the inverse element of a and * denotes the binary operation. These conditions may seem random but it's worth recognising that based on such simple axioms an entire branch of mathematics is birthed. It may come as a surprise to you that despite the multitude of "things" that can be assigned as a group, mathematicians have managed to classify every single type of group that could ever exist. But how is this possible? To take number theory as an example, its building blocks consist of the prime numbers and as such, every single number can be expressed as a product of them. This isn't too dissimilar from group theory as a special type of "building block" group exists: simple groups. The key distinction here is that there are finite categories of them (not a finite number) and it is also worth nothing that in this context we are referring to finite groups e.g the set of integers under addition would not count since it is infinite. Below is the "periodic table" of finite simple groups:



To truly appreciate how remarkable this table is, it is worth briefly examining the history behind the quest to classify all of such simple groups. It began in the early 1800s when Évariste Galois, arguably the founder of group theory, discovered the first of the simple groups and ended nearly 200 years later in 2004. By compiling roughly 500 papers from more than 100 mathematicians spanning more than 15,000 pages, it was proved that the groups that had been discovered were all the ones that could possibly exist. However, as observable from the table, there are 26 groups that are distinct and don't fit in to any of the 18 top columns. Those 18 are known as the Lie groups and the other 26 are aptly named the sporadic groups.

To explain a bit more about Lie groups and give an example, they are a special kind of group in that they are also manifolds. An object is said to be a manifold if locally it looks like a subset of the real numbers or in other words it is locally Euclidean. A concrete example of this would be the Earth, of which the set of all points on the Earth constitutes a manifold, where we accept that while it is "globally" curved in nature, "locally" for a small enough area it is flat and we can hence assign a patch of coordinates to describe this particular subset of the globe. This to say that we can map our small area to the real numbers and quite fittingly the collection of all these maps is referred to as the atlas. Below is a demonstration of such a coordinate patch:



An example of a Lie group would be the set of all n x n matrices with non-zero determinant (which implies that they are invertible and hence satisfy the inverse element condition of a group) denoted by GL(n, $\mathbb{R}$). This is known as the general linear group. Our binary operation in this case is standard matrix multiplication (which we know is associative) and we can see that the elements are closed under it along with there being an obvious identity element.
If we now consider the sporadic groups, one stands out in particular in the bottom right of the table. This happens to be what is called the monster group (sometimes referred to as the Fischer-Griess group) and it is the largest simple sporadic group that exists. A group having a large order isn't very special considering the infinite family of groups of factorial order (e.g our previous example of 3! could easily be replaced with 10!) but nevertheless, the monster's order written out in full is a sight to behold:

$$808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000$$

Cycling back to groups being the representation of the symmetries of an object, we recognise that there must be some object for which the monster group does the same. This object, mathematicians have discovered, lives in but isn't limited to 196883 dimensions. In what is known as the monster "character table" we see that in fact that it could be represented also in 21296876 and in 842609426 dimensions (and many many more). Next we have the baby monster group which can be represented in 4371 dimensions and is of order : 4,154,781,481,226,426,191,177,580,544,000,000. 20 out of the 26 sporadic groups in fact live within the monster group (more specifically they are subquotients of the monster) with this collection being named "the happy family" whereas the other 6 (the 3 Janko groups, the Rudvalis, O'Nan and Lyons group) are known as the "pariahs".

So why do these exist? Well in short, nobody really knows the answer which is deeply unsatisfying. The good news however is that sporadic groups such as the monster appear to be intimately related to modular functions which are a subset of a special type of functions called modular forms. But what is a modular form and why do we care about them? A modular form is described as a function that maps the upper half space of the complex numbers (i.e complex numbers which have a positive imaginary part) to the complex numbers under the below conditions:

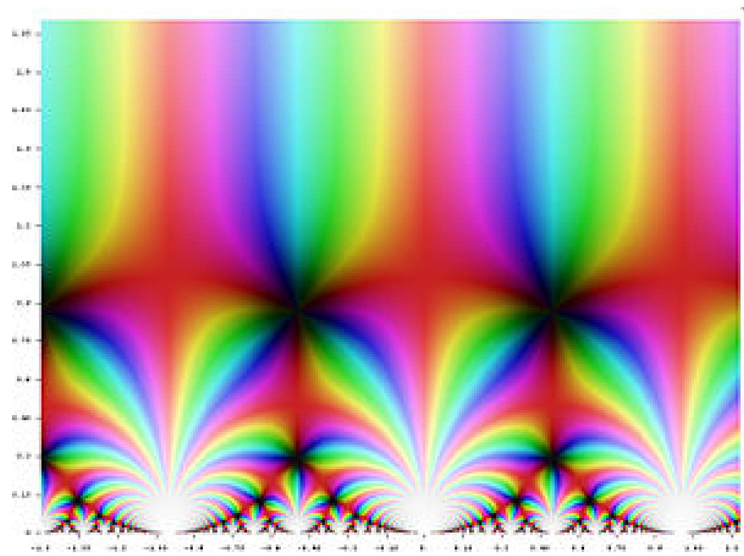
$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$$

SL_2 denotes the special linear group which is almost identical to the general linear group besides the fact that the matrices must have determinant 1 instead of simply being non-zero. Modular forms in fact were used alongside elliptic curves to prove Fermat's Last Theorem and furthermore they were used by Maryna Viazovska to prove, quite bizarrely, that the agreed upon best sphere packing method for an 8 dimensional space was in fact the most optimal or, in other words, packed them in the densest possible way. A modular function can be described as the case in which our exponent k equals 0 and hence the $(c\tau + d)$ term cancels out. The specific modular function that we're concerned with is Felix Klein's (inventor of the infamous Klein bottle) j function which has the below power series expansion along with a visual representation of the function in the complex plane:

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3...$$

$$q = e^{2\pi i}$$



John McKay was the first to make the observation that the coefficient of q is one more than the dimensions required to represent the monster group and thus the moonshine theory was born. It was first met with skepticism since it was initially marked down as a happenstance. However, after recognising that each coefficient of the powers of q could be written as the sum of the possible dimensions of the monster, it was clear that it wasn't a mere coincidence.

$$196884 = 196883 + 1$$

$$21493760 = 21296876 + 196883 + 1$$

$$864299970 = 842609326 + 21296876 + 196883 + 196883 + 1 + 1$$

Richard Borcherds shed some light on this uncanny connection in 1992 where he proved a conjecture pertaining to the relation outlined by John Conway and Simon P. Norton in their 1979 paper "Monstrous moonshine".

Often we expect logical reasoning and proof to yield "nice" values and results but this is rarely the case. The existence of the sporadic groups and the monster has undoubtedly demonstrated the sometimes lackluster and confusing nature of maths while simultaneously showing the hidden and unexpected connections between two areas of maths that could not be more different. It is almost as if we've discovered islands of mathematics and the ultimate goal is to collate them, however distant they may seem, into one singular entity. As Henri Poincaré so eloquently expressed: "mathematics is the art of giving the same name to different things".